Describing Games

- **Economics** - two areas
    - **Optimization**
    - **Equilibrium** - two areas
        - **Competitive** - don't care about what competitors are doing; participants only need to know their own technology (firms) or preferences (consumers) and the market price
        - **Interaction** - have to worry about other players; e.g., Coke vs. Pepsi, schools competing for graduate students
- **Optimization** - maximize objective subject to constraint(s); one time event
- **Game** - set of simultaneous, interrelated optimizations; choice of 1 player affects optimization of the other

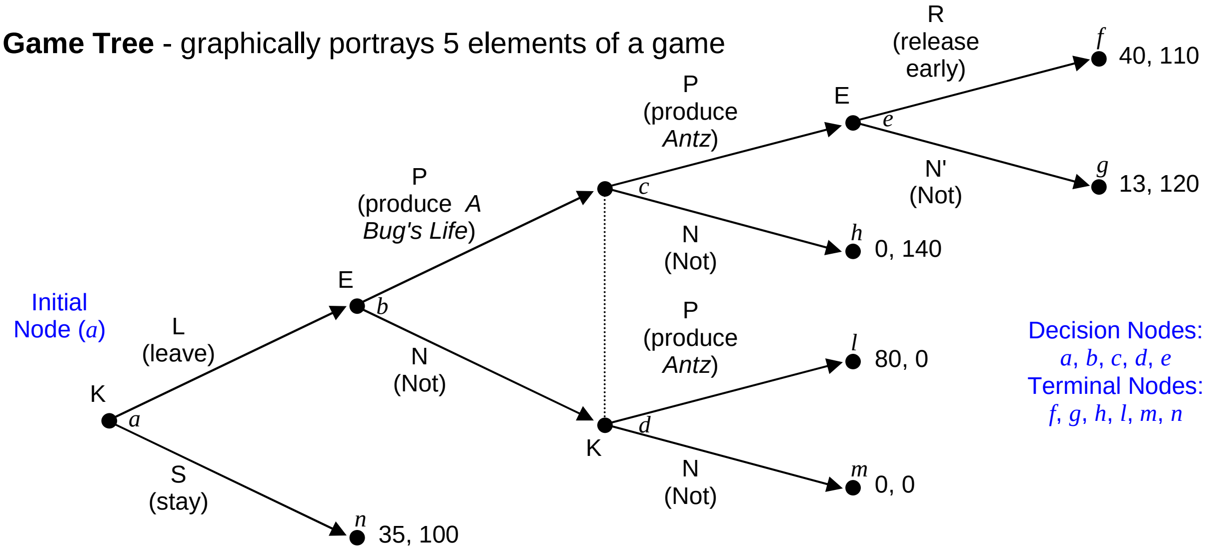
Background Info

- **Interdependence** - one person's behavior affects another person's well-being, either positively or negatively
- **Strategic Setting** - situations of interdependence; in order for one person to decide how best to behave, he must consider how others around him choose their actions
- **Purpose of Game Theory** - 2 views
    - **Normative** - help participants know what to do; "here's how you should play this game"
    - **Positive** - develop an understanding of how people actually behave; predictive theory; this view is used more for economics and social sciences
        - **Limited Rationality** - trying new models with limitations on rationality to get model predictions to better reflect real world outcomes (mainly focused on limited memory)
- **Game** - situation in which 2 or more adversaries match wits; inherently entail interdependence; usually have sets of rules that must be followed by the players
    - **Constant (Zero) Sum Game** - if one participant gains, the other loses by same amount; will always have "efficient" outcome because sum of payoffs is always the same; unrealistic; introduced by Von Neumann & Morgenstern
    - **Non-Constant Sum Game** - possible for both parties to gain (or lose); e.g., in labor strike both sides lose; brings up question of efficiency
    - **Non-cooperative Game** - participants don't work together; each player decides on his own, independent of the other people present in the strategic environment; Nash focused on non-constant sum, non-cooperative games
    - **Cooperative Game** - look at what a coalition can do, how it will form, and how it will divide profits; no all that useful because there are too many equilibria
        - **Coalition** - 2 or more players join to improve their payoffs at the expense of other players
    - **Complete Information** - participants know everything there is to know about the game (who makes what decisions and when); focuses on *structure* of the game
    - **Incomplete (Private) Information** - player knows more about something in the game than another player
    - **Perfect Information** - players knows everything that happened before (i.e., aware of previous decisions by other players); equivalent to saying all information sets have only 1 node or saying it's a sequential game; focuses on *decisions* in the game
        - **Perfect Recall** - player remembers his own choices
    - **Imperfect Information** - player doesn’t know what choice opponent made; equivalent to having a simultaneous choice or saying at least one information set has 2 or more nodes

  - **Common Knowledge** - each player knows the other has complete info
  - **1-Shot vs. Infinitely Repeated** - results depend on whether there is an infinite time horizon
  - **Chess Example** - chess is a 1-shot constant sum, non-cooperative game with complete, perfect information (except for limits on skill, calculation, and mistakes)
- **Information** - note that what's available to the modeler may not be the same as the players; psychology of players or technical aspects of firms may not be known to the modeler, but may be somewhat known by players
- **Elements of a Game** -
  - **Players** - need a list of everyone involved; 2 types
    - **Strategic Player** - makes choices
    - **Nature** - no objectives or payoffs; makes random moves
  - **Possible Actions** - complete description of what players can do; usually conditioned on where they are in the game
  - **Information** - description of what players know at each decision point (perfect vs. imperfect info, perfect vs. imperfect recall, beliefs about other players, etc.)
  - **Outcomes** - results of every possible combination of player actions
  - **Preferences** - players preferences over outcomes
    - **Ordinal** - just shows order of preference
    - **Cardinal** - shows order of preference and assigns numerical values to know how much more one outcome is preferred; required for using nature because there will be probability distributions
- **Strategy** - set of instructions on how to play the game

Extensive Form

Game Tree - graphically portrays 5 elements of a game



- **Node** - represents a place where something happens in the game
  - **Decision Node** - a player makes a decision at that place in the game
  - **Initial Node** - every extensive-form game has exactly one initial node
  - **Terminal Node** - places where the game ends; represent outcomes of the game; each terminal node corresponds to a unique path through the tree
    - **Payoffs** - listed as a vector at each terminal node; entries correspond to player order (e.g., from node *n* above, K gets 35 and E gets 100); could also use utilities

Label Them - each node is assigned to a player by putting the player number (or name) next to the node

Player 0 - nature; other players assigned numbers (1, 2, etc.)

Branch - indicates various actions that players can choose at a node

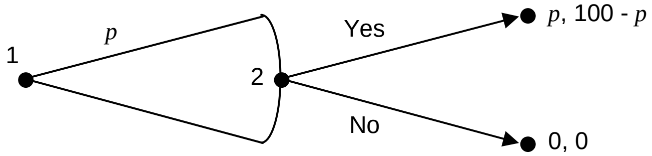
Label Them - write out description of action taken; if you want to abbreviate it make sure you use a unique identifier; note in the example Eisner (player E) has N and N' to distinguish between his two "Not" alternatives; Katzenberg (player K) however, has the same N because nodes *c* and *d* are in the same information set (simultaneous move)

Information Set - what a player knows at a decision node; every node is in one information set, although one information set can contain multiple nodes; only one decision is made at each information set

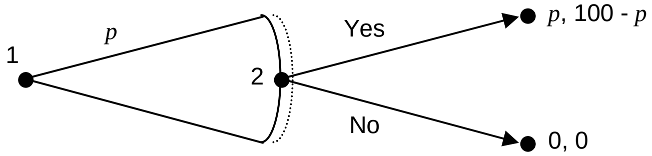
Sequential Move - player knows what opponent did prior to making his decision

Simultaneous Move - player doesn't know what opponent did prior to making his decision so the decision nodes are in the same information set; represented by connecting nodes with a dotted line; **Note:** nodes representing a simultaneous decision must have the same possible actions (see nodes *c* & *d* above)

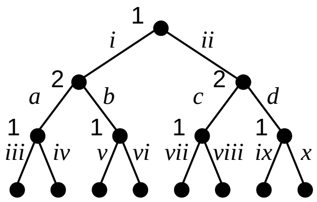
Infinite Number of Actions - represented as range; example: <u>ultimatum bargaining</u>; player 1 offers one time take-it-or-leave-it offer of anything from 0 to p dollars to sell a painting; player 2 gets a chance to accept or reject the offer; the painting is worth nothing to player 1 and $100 to player 2



Strategy - complete contingent plan for a player in the game; full specification of a player's behavior which describes the actions that the player would take at each of his possible decision points; entries in brackets denote which decision the player should make based on the opponent's previous decision (described in more detail in next section)



Example - player 1 has 8 strategies: {(*i*,[*iii*,*v*]), (*i*,[*iii*,*vi*]), (*i*,[*iv*,*v*]), (*i*,[*iv*,*vi*]), (*ii*,[*vii*,*ix*]), (*ii*,[*vii*,*x*]), (*ii*,[*viii*,*ix*]), (*ii*,[*viii*,*x*])}; player 2 has 4 strategies: {([*a*,*c*]), ([*a*,*d*]), ([*b*,*c*]), ([*b*,*d*])}



Book Version - doesn't eliminate strategies that can be ruled out such as (*i*,[*vii*,*ix*]) so player 1 has 16 strategies

Not Observable - we observe single iteration of a game at a time; that only reveals part of a player's strategy; can't observe the complete plan

"Simple Game" - tic-tac-toe; # of strategies for player 1 is between $9(7^{8})(5^{8\cdot 6})$ and $9(7^{8})(5^{8\cdot 6})(3^{8\cdot 6\cdot 4})$; for player 2 it's between $8^{9}(6^{9\cdot 7})$ and $8^{9}(6^{9\cdot 7})(4^{9\cdot 7\cdot 5})(2^{9\cdot 7\cdot 5\cdot 3})$; these numbers can be reduced if you take advantage of the symmetry of the game, but the point is it's not a complicated game to play, but it's definitely complicated to model; a person can quickly figure out how to play to a tie every time without using extensive form

Normal (Strategic) Form

Lists all strategies available to player; shows payoffs for each combination of strategies

Set Notation - normal form consists of set of players, strategy spaces for the players, and payoff functions for the players; only use set notation if too many players (or infinite strategies)

$\boldsymbol{I}$ (or S in text) - set of players (1, 2, ..., n)

$\boldsymbol{S}$ (or T in text)- set of strategies S^1 x S^2 x ... x S^n (Cartesian product; e.g., S^1 = {A,B} and S^2 = {X,Y}, $S = S^1$ x S^2 = {(A,X), (A,Y), (B,X), B,Y)})

Strategy Space, $\boldsymbol{S^i}$ - the set of all possible strategies for player i (e.g., S^1 = {A,B})

Specific Strategy, $\boldsymbol{s_i}$ - a specific strategy for player i, $s_i \in S^i$ (e.g., s_i = A)

Opponents' Strategies, $\boldsymbol{s_{\sim i}}$ - strategies played by player i's opponents (e.g., $s_{\sim i}$ = (X,D), where S^2 = {X,Y} and S^3 = {C,D}); Note: can use ~i or -i

Strategy Profile - vector of strategies, one for each player that fully describes how the game is played and is associated with a payoff vector (e.g., (A,X,D) means player 1 plays A, player 2 plays X, and player 3 plays D)

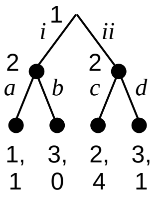
Independence - s_i is assumed to be independent of what other players do (i.e., $s_{\sim i}$); dependencies are already built in to the game (e.g., cattleman vs. farmer; cattleman has several options when there is a fence or there isn't a fence [the farmer's strategies]; the actual strategy the cattleman picks may depends on the farmer's strategy, but the strategy space is unchnaged)

$\boldsymbol{U}$ (or P in text)- set of payoff functions (u_1, u_2, ..., u_n)

Payoff Function, $\boldsymbol{u_i}$ - function with domain in set of strategy profiles (S) and whose range is the real numbers ($u_i : S \rightarrow \mathbf{R}$); sometimes written as function of strategies $u_i(s_1, s_2,..., s_n)$ or $u_i(s_i,s_{\sim i})$

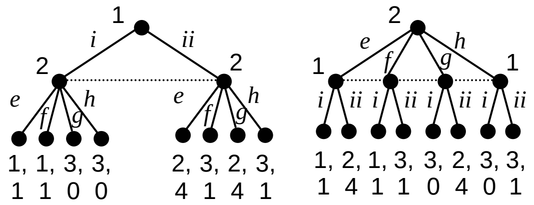
Bimatrix Game - for two player game with finite number of strategies, use matrix to list 1 player's strategies by row and the other player's by column; cells contain payoffs for each player resulting from strategies (hence name bimatrix for pairs of numbers)

Link to Extensive Form - strategic form models players that simultaneously and independently selecting complete contingent plans (strategies) for an extensive form game; there is only one strategic form for an extensive form game, but reverse isn't true (see below)



Player 2

		***ac* (*e*)**	***ad* (*f*)**	***bc* (*g*)**	***bd* (*h*)**
Player 1	***i***	1, 1	1, 1	3. 0	3, 0
	ii	2, 4	3, 1	2, 4	3, 1

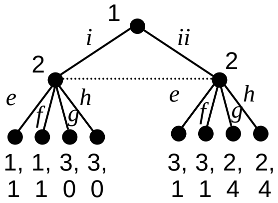


Extensive form leads to this strategic form

But this strategic form could also come from these extensive forms

Difference? - if game only has simultaneous and independent moves, strategic and extensive forms are identical; some theorists argue there is a difference, but others say there are a series of transformation s that can link all the different extensive forms that create a single strategic form; debate centers on whether these transformations change the way the game is played (i.e., are the extensive forms equivalent)

Redundant Strategy - alternatives that have same payoffs; these can be added or deleted as one of the transformations discussed above; in example shown here e & f are redundant and so are g & h



3 Player - write a matrix for each of player 3's strategies; player 3 picks the matrix to be played on and players 1 & 2 play on that matrix

4 Player - write a page of matrices; one page for each of player 4's strategies; gets difficult to visualize and not very useful

Classic Normal-Form Games

Can gain great insights from simple 2x2 games

		Player 2	
		C	**N**
Player 1	**C**	4, 4	0, 9
	N	9, 0	1, 1

Prisoners' Dilemma - two suspects are suspected of having committed a major crime, but the prosecutor only has enough evidence to convict on a lesser offense (1 year max); prosecutor needs confession (C) in order to convict for longer sentence; if one prisoner confesses, he gets a "good deal" (either 0 times or 4 years if both confess); note that payoffs equal jail time (a bad) so objective is to minimize the payoff; from perspective of an individual prisoner, it's always best to confess (dominant strategy), but if both prisoners don't confess they're better off; there's an inefficient outcome from prisoner's point of view

- **Mechanism Design** - try to set up payoffs to induce people to behave a certain way
- **Powerful Payoffs** - doesn't matter if prisoners are in separate rooms or even if they talk to each other; basic problem still exists: even if they agree to not confess, their incentives will be contrary to the agreement and they are more likely to confess than not
- **Other Examples** - donations to common-use good (free-rider problem); firms colluding
- **Solving Dilemma** - organized crime essentially changes the payoffs by punishing those who confess; trying to negate the mechanism design
- **Fundamental Insights in Economics** - only 2
  - **Invisible Hand** - in surprising ways, individuals looking for own best interest (maximizing own utility), creates efficient outcome (Adam Smith)
  - **Opposite Result** - circumstances like prisoner's dilemma where inefficiency results when people look at own best interest

		Player 2	
		R	**L**
Player 1	**R**	1, 1	0, 0
	L	0, 0	1, 1

Coordination Game - both players obtain same positive payoff if they select the same strategy, otherwise they get nothing; have multiple equilibria in which neither player has an incentive to change strategies

- **Island Example** - two drivers on opposite ends of same road have choice to drive on left or right side of road
- **Problem** - how do you get to the equilibria?
- **Communication** - players can discuss which side they will drive on; talk only works if it coincides with incentives (which is why it doesn't work in prisoners' dilemma)
- **Role of Government** - (one of many) solves coordination problem by supplying communication (tells people what side of the road to drive on)

		Player 2	
		R	**L**
Player 1	**R**	2, 2	0, 0
	L	0, 0	1, 1

Pareto Coordination - same thing by both players prefer to coordinate on a particular strategy

		Player 2	
		Bal	**Box**
Player 1	**Bal**	1, 2	-1, -1
	Box	0, 0	2, 1

Battle of the Sexes - two friends prefer to do something together but each likes one activity more than the other (here player 1 prefers boxing and player 2 prefers ballet); players have to make decision independently and simultaneously because they can't communicate

Distributional Consideration - just like coordination game, there are two equilibria, but this solution isn't as simple as arbitrarily picking one of them because payoffs are different

Matching Pennies - two players simultaneously and independently select heads (H) or tails (T) by uncovering a penny in his hand; if selections match, player 2 gives his penny to player 1; otherwise, player 1 gives his penny to player 2

Two Representations - can look at change in pennies help (top matrix) or total pennies at end of round (bottom matrix); result is the same

No Equilibrium - there isn't a cell where both players are content (at least 1 has an incentive to move to another cell)

Result - best strategy is mixed strategy; players must randomize decision so opponent won't know what the other is doing

Another Example - rock, paper, scissors game

Player 2

		H	T
Player 1	**H**	1, -1	-1, 1
	T	-1, 1	1, -1

Player 2

		H	T
Player 1	**H**	2, 0	0, 2
	T	0, 2	2, 0

Player 2

		R	P	S
Player 1	**R**	0, 0	-1, 1	1, -1
	P	1, -1	0, 0	-1, 1
	S	-1, 1	1, -1	0, 0

Uncertainty

Belief - player's belief is a probability distribution over the strategies of his opponents (μ_{-i})
Mixed Strategy - act of selecting a strategy according to a probability distribution (σ_{-i})
Pure Strategy - regular strategy (i.e., pick strategy with probability one)
Expected Payoff - "average" payoff that player would get if he played strategy si and opponents played according to μ_{-i})

$u_i(s_i,\mu_{-i}) = \sum_{s_{-i}\in S_{-i}} \mu_{-i}(s_{-i})\cdot u_i(s_i,s_{-i})$ (probability of s_{-i} times payoff to player s_i and s_{-i})

Example - player 1 believes with probability 1/2 that player 2 will play strategy L, with probability 1/3 that player 2 will play M, and probability 1/4 that player 2 will play R (i.e., μ_2(L) = 1/2, μ_2(M) = 1/4, μ_2(R) = 1/4); if player 1 selects strategy U, then the expected payoff is u_1(U,μ_2) = (1/2)8 + (1/4)0 + (1/4)4 = 5

		Player 2		
		L	M	R
Player 1	U	8, 1	0, 2	4, 0
	C	3, 3	1, 2	0, 0
	D	5, 0	2, 3	8, 1

Maximize Monetary Gain - if this is a player's objective, he is risk neutral; allows us to use monetary amounts instead of utility numbers; can also add a constant or multiply payoffs by a positive number without affecting the player's preferences (explained in more detail below)

Expected Utility Theory

Endogenous Uncertainty - results from way players play the game (e.g., mixed strategy)
Exogenous Uncertainty - not related to player actions (e.g., weather)
Expected Utility Theory - people behave as if to maximize expected utility; some people don't like the technique, but it's still used because (1) no better alternatives, (2) easy to work with; proposed by Von Neumann-Morgenstern

Lotteries - specifies payoffs and probability of receiving each payoff; anything with uncertainty can be viewed as a lottery

Simple Lottery - $L(A_1, A_2, p)$ means you get payoff A_1 (either money or a commodity bundle) with probability p and payoff A_2 with probability (1 - p); **Note:** If there are n payoffs, there will be n - 1 probabilities because the last probability is one minus the sum of the others

Compound Lottery - lottery of lotteries; $L(L_1, L_2, p)$; can always write as a simple lottery

Example - $L(L_1, L_2, {}^1/_3)$ with L_1(100, 50, $^1/_2$) and L_2(200, 10, $^1/_4$); to figure out simple lottery work out the probability of getting every possible payoff; the only way to get 200 is to get L_2 (probability 1 - $^1/_3$) and then get 200 (probability $^1/_4$) so probability of 200 is (1 - $^1/_3$)($^1/_4$) = 1/6; continue and you get L_s(200, 100, 50, 10, $^1/_6$, $^1/_6$, $^1/_6$)

Assumptions of VN-M EUT - people still debate the reasonableness of these assumptions

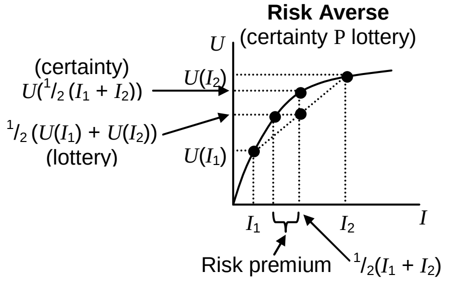
1. Preferences are **complete** and **transitive** - this means given any number of options, the player can make a choice; for any two options, the player either prefers one to the other or is indifferent between the two

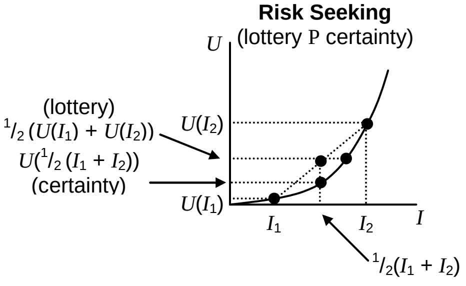
2. **Compound Lottery Assumption** - individuals are indifferent between a compound lottery and a simple lottery with the same payoffs and probabilities
    - **Contradiction** - people with pure love of gambling prefer compound lottery; people get benefit from watching the wheel spin not just from the payoff
    - **Counter** - hard to measure the satisfaction from gambling; economics focuses on outcomes not the process; love of gambling doesn't apply for all scenarios (e.g., gambling with bads... chance to lose home or life)
3. **Monotonicity** - (a) if you take a lottery and increase either or both payoffs with the same p, a player will prefer this over the original lottery (i.e., $L(A_1', A_2', p)\ \mathrm{P}\ L(A_1, A_2, p)$ if $A_1' \geq A_1$ and $A_2' \geq A_2$ and one of these is be strictly >); (b) if you take a lottery and increase the probability of the higher payoff, a player will prefer it over the original lottery (i.e., $L(A_1, A_2, p_2)\ \mathrm{P}\ L(A_2, A_2, p_1)$ if $p_2 > p_1$)
4. **Substitution** - if $L_1\ \mathrm{I}\ L_2$ then L_1 can always replace L_2 in any compound lottery and not change preferences (i.e., $L(L_2, L_3, p)\ \mathrm{I}\ L(L_1, L_3, p)$ if $L_1\ \mathrm{I}\ L_2$)
    - **Complementarities** - could argue this doesn't make sense because of complimentary goods; you may be indifferent between an apple and an orange, but when it comes time to bake an apple pie, you're not indifferent anymore
    - **Response** - that type of complementarily is not an issue in this lottery; you get L_2 or L_3, not both so you don't consume them together
    - **Sure Thing Principle** - more powerful critique; people prefer uncertainty; look at special case were $A_1\ \mathrm{I}\ A_2$, but $L(A_1, A_1, p)\ \mathrm{P}\ L(A_2, A_1, p)$ because the first one is certain and the second involves uncertainty
5. **Continuity** - if $A_1\ \mathrm{P}\ A_2\ \mathrm{P}\ A_3$ then there exists a probability q such that $L(A_1, A_3, q)\ \mathrm{I}\ A2$
    - "Proof": if $q = 1$, $L(A_1, A_3, 1) = A_1\ \mathrm{P}\ A_2$; if $q = 0$, $A_2\ \mathrm{P}\ L(A_1, A_3, 0) = A_3$; since the preference between the lottery and the certain outcome switch, somewhere in between they must be indifferent
    - **WW II Example** - two options: (1) fly 100 planes and lost 10, (2) fly 10 planes and lost all of them; same outcome either way (mission is accomplished and 10 planes lost); pilots voted and showed preference for 100 planes; in this case, people preferred uncertainty ∴ extreme example (life/death) could violate continuity

Theorem (U Exists) - there exists a utility function over certain outcomes (e.g., A_1, A_2, etc.) such that the utility of lotteries ($L(A_1, A_2, p)$) is the expected value of U:

$$U(L) = pU(A_1) + (1 - p)U(A_2)$$

- **Fair Gamble** - expectation of gains and losses = 0 (e.g., win \$1 or lose \$1 with p = 0.5)
- **Risk Neutral** - indifferent between amount with certainty and a lottery who's expected payoff is equal to the certain amount (i.e., $^1/_2(I_1 + I_2)\ \mathrm{I}\ ^1/_2(U(I_1) + U(I_2))$); utility function is linear
- **Risk Averse** - would prefer amount with certainty over lottery; utility function is concave
    - **Risk Premium** - difference between certain payoff and expected value of a lottery such that risk averse person is indifferent between the lottery and certain payoff
- **Risk Seeking** - would prefer the lottery; utility function is convex


Risk Averse
(certainty P lottery)


Risk Seeking
(lottery P certainty)

Theorem (Many U's) - the utility function will be unique up to linear transformation

$V = aU + b$ with $a > 0$

Normalizing - can use linear transformation to get $U(A_1) = 1$ and $U(A_2) = 0$ in a lottery $L(A_1, A_2, p)$ (assuming $A_1 > A_2$)

Proof: $U(A_1) = u_1 > U(A_2) = u_2$

$v_1 = au_1 + b = 1$ and $v_2 = au_2 + b = 0$... 2 equations & 2 unknowns (a & b)

Zero vs. Constant Sum Game - same thing because you can transform a constant sum game to a zero sum game

Allais' Critique of VN-M EUT - Allais was Nobel laureate who opposed Von Neumann-Morgenstern theory; conducted experiment

1. choose between L_1 = \$50K and L_2(\$250K, \$50K, \$0, 0.89, 0.1)
2. choose between L_3(\$50K, \$0, 0.9) and L_4(\$250K, 0, 0.89)

	L_3	L_4
L_1	1	2
L_2	3	4

Rational? - what happens if people fall in box 2 (similar result for 3)?

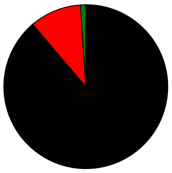
L_1 P $L_2 \Rightarrow U(50) > 0.89U(250) + 0.1U(50) + 0.01U(0) \Rightarrow$
$0.9U(50) > 0.89U(250) + 0.01U(0)$

L_4 P $L_3 \Rightarrow 0.89U(250) + 0.11U(0) > 0.9U(50) + 0.1U(0) \Rightarrow$
$0.9U(50) < 0.89U(250) + 0.01U(0)$

Opposite results! Von Neumann & Morgenstern would say these people are unreasonable

Allais' Explanation - it's OK to have L_1 P L_2 and L_4 P L_3; in the first case, the person is risk averse to the point that certainty is better (even with a much lower expected payoff); in the second case, there's not much difference in the probabilities so it's not a big deal to go for the higher payoff

Savage's Rebuttal - people make decisions based on beliefs of probability because real probabilities aren't known; said people in boxes 2 and 3 are irrational and he only needed a couple of minutes to talk with them to set them straight; rewrite lotteries and present them with a dart board; choice should only depend on where the lotteries are different (black and green areas only)



	Black	Green	Red
L_1	50K	50K	50K
L_2	250K	0	50K
L_3	50K	50K	0
L_4	250K	0	0

Irrelevant so L_1 I L_3 and L_2 I L_4

??? Test - went out and performed Allais' experiment; then gave a card with Allais explanation to the people who picked boxes 1 and 4 and a card with Savage's explanation to the people who picked boxes 2 and 3; had more people switch from Allais' argument rather than Savage's

Analyzing Games

Strategic Tensions - listed throughout book, summarized here:

- **First** - clash between individual and group interests (prisoners' dilemma); can be resolved with commitment devices (threats, legally binding contracts, etc.) which will change payoffs or other part of the strategic setting
- **Second** - strategic uncertainty; rationalizability doesn't always lead to a unique strategy profile; even if it does, rationalizability only requires players' behavior and beliefs be consistent with rationality, which doesn't mean the beliefs are correct; can be resolved through institutions, rules, norms of behavior and culture that facilitate coordination in society
- **Third** - inefficient coordination; Nash equilibrium (so neither side has a unilateral incentive to deviate) that is not Pareto optimal

Strictly Ordered Game - clear ranking among alternatives; lots of different ways to set this up (72), but number of insights are limited

Three Basic Insights -

- Dominant Strategy for 1 (or both) players
- Multiple Equilibria -
- No Equilibrium -

Static Games - all players' actions are taken simultaneously and independently; also called one-shot games

Player 2

Player 1

	A	B
A	1, 1	2, 2
B	3, 3	4, 4

Dominance

Strictly Dominated Strategy - something else is better in every contingency (every opponent's strategy); A2 is strictly dominated by A2 if payoff for A1 > payoff for A2 for all opponent's strategies B1, B2,..., Bn

- **Weakly Dominated** - same as strictly, but can have some payoffs = (others are >)
- **Rationality Requirement** - minimum requirement for player to be rational is to not play a strictly dominated strategy
- **Pure Strategy Domination** - one strategy is better than another
- **Mixed Strategy Domination** - combination of strategies is better than the dominated strategy; harder to find because probabilities don't have to be 50/50 and could be more than just two strategies; trick is to look for alternating patterns of large and small numbers in payoff matrix
  - **Example** - play A_1 half the time and A_2 half the time; expected payoff for player 1 is 0.5(10) + 0.5(0) = 5 for both of player 2's strategies; that means A_3 is strictly dominated
- **Formally** - $\exists\, p$ such that $px_i + (1 - p)y_i > z_i$ for $i = 1, 2, \ldots, n$ (all of opponent's strategies)
  - **When Exist?**

    If both > 0, A_1 dominates A_2

    If both < 0, A_2 dominates A_1

    $px_1 + (1 - p)y_1 > z_1 \Rightarrow p(x_1 - y_1) > z_1 - y_1$

    $px_2 + (1 - p)y_2 > z_2 \Rightarrow p(x_2 - y_2) > z_2 - y_2$

    Assume $x_1 > y_1$ and $x_2 < y_2$: (works other way too):

Player 2

Player 1

	B_1	B_2
A_1	10, b_{11}	0, b_{12}
A_2	0, b_{21}	10, b_{22}
A_3	4, b_{31}	4, b_{32}

Player 2

Player 1

	B_1	B_2
A_1	x_1, b_{11}	x_2, b_{12}
A_2	y_1, b_{21}	y_2, b_{22}
A_3	z_1, b_{31}	z_2, b_{32}

$$p > \frac{z_1 - y_1}{x_1 - y_1} \text{ and } p < \frac{z_2 - y_2}{x_2 - y_2}$$

$\therefore$ if $\frac{z_1 - y_1}{x_1 - y_1} < \frac{z_2 - y_2}{x_2 - y_2}$ a mixed strategy (p for A_1 and (1 - p) for A_2) dominates A_3

Expectations - have to be taken with utilities, not payoffs; problem with that is payoffs ($) are observable and utilities aren't; solution is to use payoffs and assume risk neutral

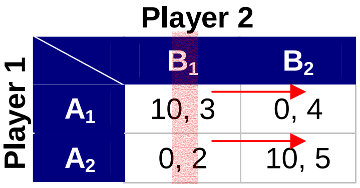
Risk Aversion - built into utility values so any unhappiness resulting from randomness is already built in (i.e., can't argue a sure (4, 4) is better than an expected (5, 5))

Principal-Agent Problem - if owner has lots of workers, it's usually OK to assume risk neutral owner

Iterated Dominance - process of eliminating strategies that are strictly dominated; go through for each player looking for a dominated strategy; if one is found for any player, go through the sub-game (with the dominated strategies removed) and look for more strictly dominated strategies; repeat until there are no more strictly dominated strategies

Simple Game - only eliminate dominated strategies

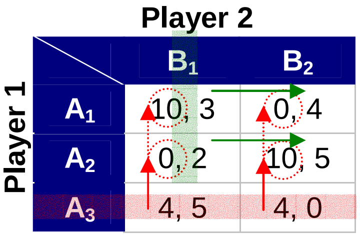
Example - no dominant strategy for player 1, but can see that player 2 has a dominant strategy (B_2); $\therefore$game is simplified to having player 1 pick between the 0 and 10 of A_1 and A_2



	B_1	B_2
A_1	10, 3	0, 4
A_2	0, 2	10, 5

Iterative -

Example - can remove last row because A_3 is strictly dominated by mixed strategy involving A_1 and A_2 (50-50 as in previous example); now look for another strategy that's strictly dominated (B_1 is by B_2)



	B_1	B_2
A_1	10, 3	0, 4
A_2	0, 2	10, 5
A_3	4, 5	4, 0

Debate - some theorists don't like iterative process because it's to complicated to look for dominated strategies

Rationalizable Strategies - set of strategies that survive iterated dominance

Efficiency

More Efficient - strategy s is more efficient than s' if all the players prefer (or are indifferent between) the outcome of s to the outcome of s' and the preference is strict for at least one player

Pareto Efficient - s is Pareto efficient (or Pareto optimal) if there is no other strategy profile that is more efficient (i.e., cannot improve payoff to any player without hurting other players)

Why Care - efficiency doesn't lead us to an equilibrium, but allows us to talk about the relative quality of multiple equilibria (or even a single one)

Best Response

What To Play - depends on beliefs of what opponent will do

Beliefs - can be decided either by dominance criteria or by probabilities; theorists argue this point; some says dominant strategy is all there is; others argue for beliefs about opponent based on the game (e.g., iterative dominated strategies)

Best Response - lists all of a player's best choices given each of the opponents' choices; response is to belief, not to opponent so theorists argue the name (reply, reaction, etc.)

Formally - $BR_i(\mu_{-i})$ or $R^i(\mu_{-i})$ is set of best responses $s_i \in S_i$ such that $u_i(s_i, \mu_{-i}) \geq u_i(s_i', \mu_{-i})$ for every $s_i' \in S_i$

Finding It - assume opponent will pick 1 strategy then find best reply; move to next strategy for opponent and pick the best reply to that; this generates a reaction function or best response

Strategic Form Only - only use best reply function for strategic form; extensive form shows order of play so there could actually be a response

Example - supposed player 1's belief is that player 2 will play L with probability 1/3, C with probability 1/2, and R with probability 1/6; look at player 1's payoffs:

$u_1(\text{U}, \mu_2)$ = (1/3)2 + (1/2)0 + (1/6)4 = 8/6

$u_1(\text{M}, \mu_2)$ = (1/3)3 + (1/2)0 + (1/6)1 = 7/6

$u_1(\text{D}, \mu_2)$ = (1/3)1 + (1/2)3 + (1/6)2 = 13/6

$\therefore BR_1$(1/3, 1/2, 1/6) = {D}

Player 2

Player 1	L	C	R
U	2, 6	0, 4	4, 4
M	3, 3	0, 0	1, 5
D	1, 1	3, 5	2, 3

Dominance vs. Best Response - for any finite game, the set of best responses for a player will be a subset of the set of strategies that are not strictly dominated; for a finite, two-player game the sets will be the same

Analysis Steps -

1. Look for strategies that are best responses to simplest beliefs (pure strategies); these strategies obviously cannot be strictly dominated
2. Look for strategies that are strictly dominated by other pure strategies
3. Look for strategies that are strictly dominated by mixed strategies

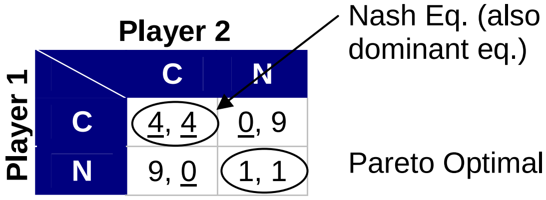
Nash Equilibrium

Nash Equilibrium - set of strategies, one for each player, $(s_1^*, s_2^*, \ldots, s_n^*)$ where $s_i^* \in S^i$ such that $\forall\ i,\ u_i(s_i^*, s_{\sim i}^*) \geq u_i(s_i, s_{\sim i}^*)\ \forall\ s_i \in S^i$

English - each payer's strategy is a best response to the best responses of the other players; no player has an incentive to change his choice

Finding It - for each player underline his best response to each of the opponents' strategies; any cell (set of strategies) that has both payoffs underlined is a Nash equilibrium

Nash ≠ Optimal - Nash equilibrium is not necessarily Pareto optimal (e.g., prisoners' dilemma)



Player 2

Player 1	C	N
C	$\underline{4}$, $\underline{4}$	$\underline{0}$, 9
N	9, $\underline{0}$	1, 1

Nash Eq. (also dominant eq.)

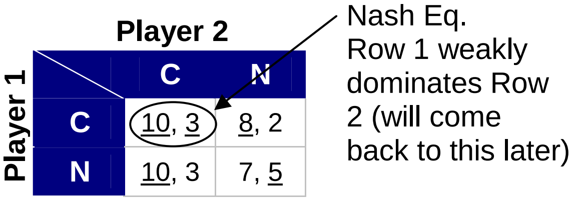
Pareto Optimal

Strong Nash Equilibrium - some theorists try to expand Nash equilibrium to look at coalitions of players not being able to improve by deviating

Problems - (1) when strong Nash equilibrium exists, problem is trivial, (2) how do players form coalitions? if side payments to get players to cooperate, that changes the game, (3) cost of coordination; if it's too high, strong Nash won't happen

Strict Nash Equilibrium - uses > instead of ≥; argument is that it can't be equilibrium is player can pick a different strategy with the same payoff

Problem - rarely exists



Player 2

Player 1	C	N
C	$\underline{10}$, $\underline{3}$	$\underline{8}$, 2
N	$\underline{10}$, 3	7, $\underline{5}$

Nash Eq.
Row 1 weakly dominates Row 2 (will come back to this later)

Infinite Strategies - if game has infinite strategy spaces, compute best response mapping (BR_i as function of opponents' strategies); solve system of equations to find where best response functions intersect (see duopoly examples or mixed-strategy Nash equilibrium below)

Nash Theorem - if (1) S^i is compact and convex for every i and U^i is (2) jointly continuous in s_i and $s_{\sim i}$ (i.e., everybody's strategies) and is (3) quasiconcave in s_i,(i.e., own strategy) then a Nash equilibrium exists

S^i Bounded - if not bounded, could have possibility of no best reply

S^i Compact - if not compact, could also have possibility of no best reply

S^i Convex & U^i Quasiconcave - best replies are convex sets

U^i Jointly Continuous - guarantees best reply changes with respect to opponent's strategy in well behaved way; appeal to fixed point theorem (shows if system of equations has solution) to show that best replies intercept

Checking 2 & 3 - find best reply functions, show they're well behaved and intersect; the intersection is the Nash equilibrium

Mixed-Strategy Nash Equilibrium

Loosen Assumptions - can still have Nash equilibrium without all these assumptions

Discrete Strategies - if S^i is discrete it isn't convex by definition, but can still have Nash equilibrium

Mixed Strategies - all games with finite strategies will have a Nash equilibrium with mixed strategies (probability distribution over strategy space)

Strategy Space - [0, 1]... this is compact (closed & bounded) and convex (line connecting any two points is still in the interval)

General Rule - given finite number of strategies, any probability distribution over those strategies will be compact and convex

Expected Payoff - $E(U^1) = apq + bp(1 - q) + c(1 - p)q + d(1 - p)(1 - q)$ $= (a - b - c + d)pq + (b - d)p + (a + c)q + d$

Continuous function of p & q; also, if we fix q, this is linear in p (which is quasiconcave)

			Player 2	
			q	$1 - q$
			H	T
Player 1	p	H	a	b
	$1 - p$	T	c	d

Mixed-Strategy Nash Equilibrium - no player can change his probabilities and do better; <u>2 key criteria</u> (solve problems [i.e., find probabilities] with the first one, then check the second one based on the probabilities found in the first one):

1. If two (or more) strategies are played with positive probability by a player, then the expected value of those strategies are equal to each other
2. The expected payoff of strategies with positive probability must be at least as great as (≥) the expected value from strategies with zero probability

			Player 2			
			q_1	q_2	q_3	q_4
			W	X	Y	Z
Player 1	p_1	A	a_1, w_1	a_2, x_1	a_3, y_1	a_4, z_1
	p_2	B	b_1, w_2	b_2, x_2	b_3, y_2	b_4, z_2
	p_3	C	c_1, w_2	c_2, x_3	c_3, y_3	c_4, z_3
	p_4	D	d_1, w_4	d_2, x_4	d_3, y_4	d_4, z_4

Assume p_1 & $p_2 > 0$; p_3 & $p_4 = 0$

$$\overbrace{\sum_{j=1}^{4} a_j q_j = \sum_{j=1}^{4} b_j q_j}^{\text{Criteria 1}} \overbrace{\geq \sum_{j=1}^{4} c_j q_j}^{\text{Criteria 2}}$$

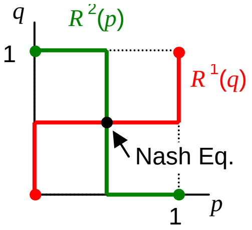
$$\sum_{j=1}^{4} a_j q_j = \sum_{j=1}^{4} b_j q_j \geq \sum_{j=1}^{4} d_j q_j$$

Weak Nash - since the criteria outlined above requires ≥ (vs. >), mixed-strategy Nash equilibrium is a form of weak Nash equilibrium (not strict equilibrium)

- **Matching Pennies** - has no Nash equilibrium; play with mixed strategy (e.g., player 1 picks H with probably p and T with probability 1 - p)
  - **Best Reply for Player 1** - look at boundary points first
    - $q = 1 \Rightarrow$ best reply is H ($p = 1$)
    - $q = 0 \Rightarrow$ best reply is T ($p = 0$)
    - **In Between** - compare expected payoffs for player 1 using H and T
      - $EV_H^1 = 1q - 1(1 - q) = 2q - 1$
      - $EV_T^1 = -1q + 1(1 - q) = 1 - 2q$
      - At q near 0, $EV_T^1 > EV_H^1$ so player 1 should play T; At q near 1, $EV_H^1 > EV_T^1$ so player 1 should play H
      - EV_T^1 and EV_H^1 converge at $q = 0.5$ at which point player 1 is indifferent between H and T
  - **Best Reply for Player 2** - similar argument to player 1
    - $p = 1 \Rightarrow$ best reply is T ($q = 0$)
    - $p = 0 \Rightarrow$ best reply is H ($q = 1$)
  - **Nash Equilibrium** - best replies intersect at $p = q = 0.5$

Player 2

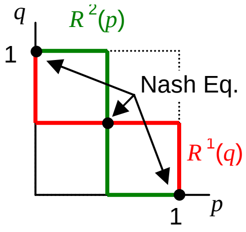
Player 1		q	$1 - q$
		H	**T**
p	**H**	$\underline{1}$, -1	-1, $\underline{1}$
$1 - p$	**T**	-1, $\underline{1}$	$\underline{1}$, -1



- **General Case** - payoffs for mixed strategies are linear in probability so will always have best replies like this (stay on one value and at some p [or q], player is indifferent)
  - **Expected Value** - find EV_k^i, expected value of player i playing strategy k
  - **What to Play** - assign probability 1 to single strategy with largest expected value; otherwise split probability equally between all strategies that tie for largest expected value
    - **Find Split** - to find where player is indifferent, set expected values equal to each other and solve for opponent's probability
  - **Coordination Game** - has two pure strategy Nash equilibria and one mixed strategy Nash equilibria

Player 2

Player 1		q	$1 - q$
		L	**R**
p	**L**	$\underline{1}$, $\underline{1}$	0, 0
$1 - p$	**R**	0, 0	$\underline{1}$, $\underline{1}$



- **Purification** - turning a mixed-strategy into a pure strategy by conditioning on some "irrelevant" piece of information not known to the opponent (or opponent knows info, but doesn't know how the info is being used); randomness introduced by nature (not by the player) so the player doesn't bear the cost of randomization
  - **Poker Example** - have a hand [3D, 3H, QC, QD, *]; if the fifth card is > 7, bet; if it's ≤ 7 don't bet

- **Debate** - some theorists doesn't like mixed-strategy equilibrium idea, but applied economists like any equilibrium because they look at an equilibrium and predict how it'll change (with comparative statics or other tool)
  - **Empirical Data** - can never confirm an equilibrium (never have enough info), but can test direction of change of an equilibrium (e.g., if we do X, what happens to the equilibrium)

Mixed-Strategy Results - comparative statics of mixed-strategy equilibria generally aren't intuitive

Player 2

Player 1		q	$1 - q$
		War	**No**
p	**War**	a_1, b_1	a_2, b_2
$1 - p$	**No**	a_3, b_3	a_4, b_4

If $a_1\uparrow$, typical intuition says player 1 is more likely to play row 1

But assume there is a mixed-strategy equilibrium for each player

Criteria 1 says:

$a_1q + a_2(1 - q) = a_3q + a_4(1 - q)$

$b_1p + b_2(1 - p) = b_3p + b_4(1 - p)$

Solving for p and q shows $q = \dfrac{a_4 - a_2}{a_1 + a_4 - a_2 - a_3}$ and $p = \dfrac{b_4 - b_2}{b_1 + b_4 - b_2 - b_3}$

$\therefore$ $a_1\uparrow \Rightarrow q\downarrow$; row player's probabilities doesn't depend on his payoffs; goal of mixed-strategy equilibrium is to make opponent indifferent so p only depends on opponent's payoffs

Critics - OK, you get non-obvious results with mixed-strategy equilibrium, but they may not useful results

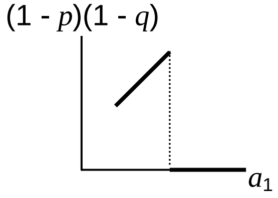
Local vs. Global Results - a local result depends on derivatives in the immediate area (e.g., stay with a mixed-strategy), but the global result could be totally different; not uncommon to have local & global comparative statics move in different directions

Equilibrium Correspondence - looks at equilibrium versus some payoff (e.g. probability of going to war $[(1 - p)(1 - q)]$ versus cost of going to way for player 1 $[a_1]$)

Local Result - stay in mixed-strategy equilibrium; $a_1\uparrow \Rightarrow$ increased chance of war (not intuitive result)

Global Result - change a_1 enough and we get global result: $a_1\uparrow \Rightarrow$ no war

Slutsky - to find local comparative statics "typically the way we do it is we differentiate the hell out of everything"



Slutsky - Chapter 9 is "reasonably incomprehensible" and congruous is a "word he just made up"

Caution - don’t be swayed by arguments based on large differences in payoffs; can be fooled into "what is reasonable", but it doesn't matter because a transformation can always change the maximum and minimum payoffs to be as close together (or far apart) as you want (based on Von Neumann-Morgenstern expected utility theory); example looking at payoffs for row player only and assuming (A1,B1) is Nash equilibrium; seems reasonable to play A2 in order to eliminate risk from irrational opponent (or mistake); after all, difference in payoff is very small (10 vs. 9.999), but difference from mistake is huge; problem is you can use a transformation (V = aU + b) to get to the less dramatic game next to it (ε is some small number); in fact, you could get the payoffs to be 0.99 instead of 0 (it'll make ε even smaller)

Player 2

Player 1		**B_1**	**B_2**	**B_3**	**B_4**
	A_1	10	-10^{10}	-10^{10}	-10^{10}
	A_2	9.999	10	10	10

Player 2

Player 1		**B_1**	**B_2**	**B_3**	**B_4**
	A_1	1	0	0	0
	A_2	1 - ε	1	1	1

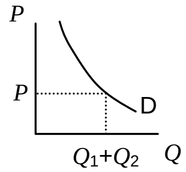
Duopoly Examples

Painful micro-esque detail on finding best response functions

Cournot Duopoly

Assumptions - decision made simultaneously; firms make identical product and compete by setting quantity (Q); take products to auction where price is determined based on total output by both firms; assume risk neutral so we can use expected profit & not worry about utility

Real World - makes sense when firms make Q decision well before output is realized (e.g., agriculture)

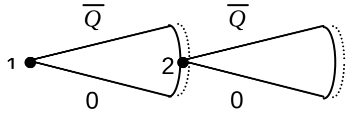


Cost - $C^i(Q_i)$; function of individual firm's output

Price - $P(Q_1 + Q_2)$; function of total output (that's where the interactoin comes in)

Profit - $\pi^i(Q_1, Q_2) = P(Q_1 + Q_2)\cdot Q_i - C^i(Q_i)$

Strategy - select $Q_1 \in [0,\overline{Q}]$ to maximize expected profit; we can always set Q large enough to avoid corner solutions



$\pi^1(Q_1, Q_2) = P(Q_1 + Q_2)\cdot Q_1 - C^1(Q_1)$
$\pi^2(Q_1, Q_2) = P(Q_1 + Q_2)\cdot Q_2 - C^2(Q_2)$

Best Reply - given belief that firm 2 will product Q_2, firm 1 will max profit by producing $Q_1 = R^1(Q_2)$, best reply function (firm 1's optimal choice); find by solving:

Max $\pi^1(Q_1, Q_2) = P(Q_1 + Q_2)\cdot Q_1 - C^1(Q_1)$... set 1st derivative = 0 and solve for Q_1

product rule

$$\frac{\partial P}{\partial Q}Q_1 + P - \frac{\partial C^1}{\partial Q_1} = 0\,;$$

Notes: $Q = Q_1 + Q_2$, by chain rule $\frac{\partial P(Q_1+Q_2)}{\partial Q_1} = \frac{\partial P}{\partial Q}\cdot\frac{\partial Q}{\partial Q_1} = \frac{\partial P}{\partial Q}\cdot 1$

$\frac{\partial P}{\partial Q} < 0$ (law of demand)

Do the same for firm 2 to get $Q_2 = R^2(Q_1)$

Interaction - to find how Q_1 varies with Q_2, take total derivative of 1st order cond wrt Q_2:

$$\underbrace{\left[\overbrace{\frac{\partial^2 P}{\partial Q^2}Q_1+\frac{\partial P}{\partial Q}}^{\text{Product rule}}+\frac{\partial P}{\partial Q}-\frac{\partial^2 C^1}{\partial Q_1^2}\right]}_{\text{Derivative wrt } Q_1}\frac{dQ_1}{dQ_2}+\underbrace{\frac{\partial^2 P}{\partial Q^2}Q_1+\frac{\partial P}{\partial Q}}_{\text{Derivative wrt } Q_2}=0$$

$$\frac{dQ_1}{dQ_2} = \frac{-\left[\frac{\partial^2 P}{\partial Q^2}Q_1+\frac{\partial P}{\partial Q}\right]}{\frac{\partial^2 P}{\partial Q^2}Q_1+\frac{\partial P}{\partial Q}+\frac{\partial P}{\partial Q}-\frac{\partial^2 C^1}{\partial Q_1^2}} = \frac{-\frac{\partial^2 \pi^1}{\partial Q_1 \partial Q_2}}{\frac{\partial^2 \pi^1}{\partial Q_1^2}}$$

Another way of looking at this is to work directly with $\pi^1(Q_1, Q_2)$

1st Order: $\frac{\partial \pi^1}{\partial Q_1} = 0$

Total derivative wrt Q_2: $\frac{\partial^2 \pi^1}{\partial Q_1^2} \cdot \frac{dQ_1}{dQ_2} + \frac{\partial^2 \pi^1}{\partial Q_1 \partial Q_2} = 0$

Assume $\frac{\partial^2 \pi^1}{\partial Q_1^2} < 0$ (sufficient 2nd order condition for maximization problem)

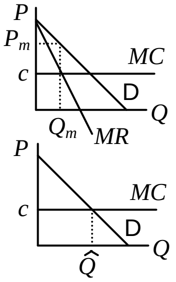
$\therefore$ $\frac{dQ_1}{dQ_2}$ has same sign as $\frac{\partial^2 \pi^1}{\partial Q_1 \partial Q_2} = \frac{\partial^2 P}{\partial Q^2} Q_1 + \frac{\partial P}{\partial Q}$; 1st term usually > 0; 2nd is < 0

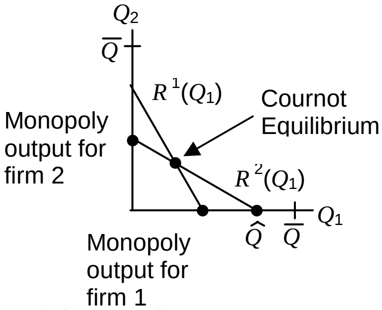
Linear Demand - to get further results we assume linear demand so

$\frac{\partial^2 P}{\partial Q^2} = 0$; that means $\frac{dQ_1}{dQ_2} = \frac{\partial P}{\partial Q} < 0$ (i.e., $Q_2\uparrow \Rightarrow Q_1\downarrow$)

Note: this is with linear assumption; in theory it's possible to have $dQ_1/dQ_2 > 0$ if demand is convex enough (i..e, $\partial^2 P/\partial Q_2 > 0$)

Constant Marginal Cost - also assuming no fixed cost and identical, constant marginal cost (c) for both firms (i.e., $C^1(Q_1) = cQ_1$)





Monopoly Output - what a firm would produced if the other firms produces zero

Competitive Output - the only way a firm can drive the opponent to produce zero output is to product at the competitive output level ($P = MC$)

Symmetry - since we assume each firm has identical marginal cost, graph of best response functions is symmetric; had we assumed otherwise, one of the two best response would be steeper (or flatter) than the other

Equilibrium - evident that equilibrium should be at the intersection of the two best response functions (neither player has an incentive to change his strategy... that's a Nash equilibrium)

Dominant Solvable - iteratively eliminating dominated strategies yeilds same result (equilibrium at intersection)

Isoprofit Curves - level curves of a firm's profit (e.g., $\pi^1(Q_1, Q_2) = c$)

$\partial\pi^1/\partial Q_2 < 0$ - because Q2↑ ⇒ P↓ and $\pi^1(Q_1, Q_2) = P(Q_1 + Q_2)\cdot Q_1 - C^1(Q_1)$

$\partial\pi^1/\partial Q_1 = 0$ - at Q_1^* (best response; i.e., any Q_1 on $R^1(Q_2)$)

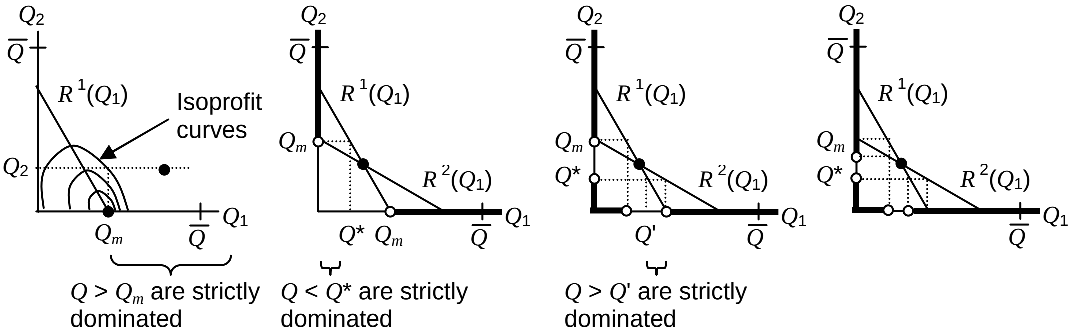
Shape - because of first order conditon shown above, curves peak on the best reply curve ($R^1(Q_2)$); for all $Q_1 < Q_1^*$, $\partial\pi^1/\partial Q_1 > 0$ (moving up to max); for all $Q_1 > Q_1^*$, $\partial\pi^1/\partial Q_1 < 0= 0$ (moving away from max); max profit occurs at Q_m (monopoly output) so isoprofit curves get better as you move down to Q_m

First Round - looking at isoprofit curves, it would never make sense to play any quantity greater than Q_m; regardless of the opponent's output, any $Q > Q_m$ will be on an inferior isoprofit curve $\therefore$ all $Q > Q_m$ are strictly dominated

Second Round - look at opponent playing Q_m; let Q^* be the firm's best response (based on the best reply curve, $R^i(Q_j)$); all $Q < Q^*$ are strictly dominated because they are on inferior isoprofit curves

Third Round - look at firm play Q^* from previous round; let Q' be the firm's best response (based on the best reply curve); all $Q > Q'$ are strictly dominated because they are on inferior isoprofit curves

More Rounds - this process continues indefinitely and gets a tighter and tighter bound on the intersection of the best reply curves; each round is more complicated to find the dominated strategies (that's why some theorists don't like iterative dominance); **Note:** didn't need the nonlinear assumption for this conclusion, just downward sloping R^1 and R^2 (but it makes the graphs easier)



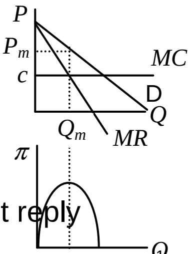
Problem - infinite iterative dominance gets to equilibrium, but that's not what people do it real life (don't have that kind of time!); this problem is fairly simple and we can look at the graph and figure out the equilibrium will be at the intersection of the best reply curves; problem comes in games that have multiple intersections; each may be a Nash equilibrium, but how do we get there?

Fictional Referee - takes player's inputs and asks them to rechoose if the result is not an equilibrium

Bertrand Duopoly

Assumptions - similar to Cournot model except firms compete on price, not quantity; extensive form looks identical (just change Q to P in the decisions); will assume same constant marginal cost (c) for both firms to make this easier; also assume firms equally split the market if they charge the same price and firms cannot collude

Real World - makes sense when firms make P decision and then produce (i.e., decide Q) based on orders (e.g., catalog sales)



Cost - $cQ^i(P_1, P_2)$; quantity is function of price of both firms

Profit - $\pi^i(P_1, P_2) = P_i \cdot Q^i(P_1, P_2) - cQ^i(P_1, P_2)$

Best Reply - three cases:

$P_1 > P_2$ - firm 1 sells nothing so $\pi^1 = 0$

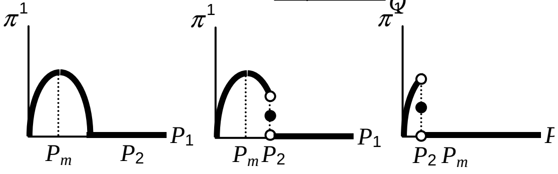
$P_1 = P_2$ - firms evenly split market so $\pi^1 = \pi^2 = \pi^1(P_1 + P_2, \infty)$ (profit if firm 1 charged price $P_1 + P_2$ and firm 2 charging more so it gets no sales)

$P_1 < P_2$ - firm 1 sells everything and profits as if it were a monopolist; firm 1's best reply depends on relationship between P_2 and P_m

$P_2 > P_m$ - firm 1 should charge P_m to get monopoly profit

$P_2 < P_m$ - firm 1 should charge slightly less than P_2

Problem - there is technically no best reply because function is not continuous at point of firm 1's best reply (wants the open dot to max profit, but at that point it drops to the solid dot)



Dominant Solvable - look at various cases to see what can be ruled out

$P_1 > P_2 > MC$ - firm 1 wouldn't do this; it could do better be charging less than P_2

$P_1 > P_2 = MC$ - firm 2 wouldn't do this; it could do better by charging more

$P_1 > MC > P_2$ - firm 2 would lose money so it wouldn't do this

Rules out any case with $P_1 \neq P_2$

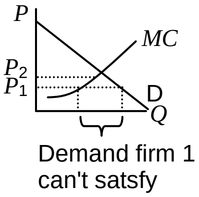
$P_1 = P_2 > MC$ - either firm will do better by lowering price

$P_1 = P_2 < MC$ - both firms losing money

$P_1 = P_2 = MC$ - neither firm can do anything to improve

Equilibrium - set of strategies each firm chooses with no incentive to change; in this case an equilibrium point is $P_1 = P_2 = MC$

Lesson - equilibrium can stil exists when best reply doesn't

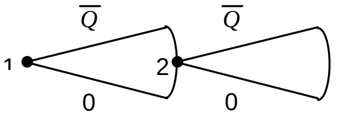


Non-Constant MC - can't rule out possibility that firm 2 sells with $P_2 > P_1$ because firm 1 may not be able to supply enter market at P_1 (see graph); this requires more complicated analysis and may find that there is no pure strategy equilibrium

Differentiated Product Model - more interesting results; may smooth out discontinuity so that best reply exists

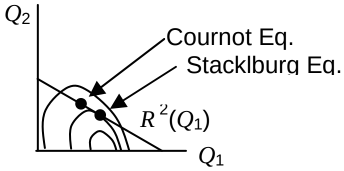
Stacklburg Duopoly

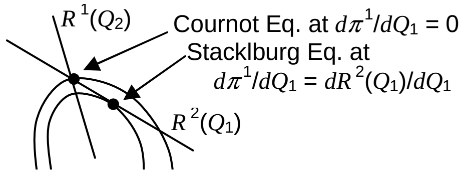
Assumptions - similar to Cournot model except one firm decides output before the other (let's say firm 1 goes first; so firm 2 knows Q1 before deciding on Q2)



$\pi^1(Q_1, Q_2) = P(Q_1 + Q_2) \cdot Q_1 - C^1(Q_1)$

$\pi^2(Q_1, Q_2) = P(Q_1 + Q_2) \cdot Q_2 - C^2(Q_2)$





Cournot Equilibrium - Q_1^C and Q_2^C are quantities produced by each firm in Cournot equilibrium; note that these quantities are the intersection of the two firms' best response curves ($R^1(Q_2)$ and $R^2(Q_1)$; each curve being where a firm maximizes profit given it's opponents output; i.e., point were $d\pi^i/dQ_i = 0$); if firm 1 produces at $Q_1 = Q_1^C$, then best reply for firm 2 is Q_2^C; ∴ the Cournot equilibrium is also an equliibrium for the Stacklburg game (but not the only one)

Other Equilibria - firm 1 knows that firm 2 will play its best response to firm 1's output; firm 1 can then view firm 2's best reponse function ($R^2(Q_1)$) as it's feasible set and try to maximize its profit; this occurs at the point were $R^2(Q_1)$ is tangent to one of firm 1's isoprofit lines (that's the best firm 1 can do); if firm 1 selects this level of output firm 2 will respond with it's best response and this is another equliibrium to the game (this is the Stacklburg equliibrium); Note: firm1 prefers this new equlibrium because it is better off than in the Cournot equlibrium

Credible Threat - firm 2 would prefer the Cournot equilibrium; it may try to get there by threatening firm 1 and saying that it will product Q_2^C regardless of what firm 1 produces in order to force firm 1 to produce Q_1^C

Subgame Perfection - once firm 1 has decided what it will do, the only logical choice for firm 2 is to procede with it's best reponse; any other action (e.g., playing Q_2^C because of a threat) would hurt firm 2; ∴ the threat to produce Q_2^C is not credible (firm 2 will not commit to something that is not optial)

Solution - firm 2 can look for some type of commitment technology to convince firm 1 that firm 2 will produce Q_2^C regardless of what firm 1 does; real world examples:

Airlines - airlines use agents with very limited bargaining power to make fares (threats) credible; otherwise, passengers would haggle over prices and based on subgame perfection, airlines would accept them

Cold War - USA and USSR operated on strategy of mutually assured descruction (if either side attacked, the opponent would respond with a massive nuclear attack that would eventually destroy both sides); problem was that this wasn't an optimal response so it wasn't a credible threat; *Dr. Stangelove* (fictional movie) talked about an automated response in order to make the threat credible

Refinements - theorists starting to incorporate probability that a player will respond irrationally (for all models, not just Cournot, Bertrand, & Stacklburg)

Total Differential vs. Total Derivative

$z = f(x, y)$

Total Differential: $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Total Derivative with respect to x: $\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$

Refinements

Refinements - change set of equilibria to find "better" set of equilibria by eliminating some that are less plausible

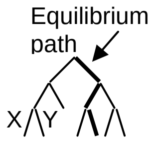
Strategic Form

Eliminate Weakly Dominated Strategies -

- **Purpose** - throwing out strictly dominated strategies (even iteratively) never eliminates a Nash equilibrium (because Nash equilibrium never has player using a strictly dominated strategy); goal is to eliminate weakly dominated strategies to get a better defined equilibrium
- **Example** - two people meeting at the airport; any place they meet is a Nash equilibrium, but there are better places to meet (e.g., ticket counter vs. 5th hangar)
- **Example** - in this game both (6, 3) and (6, 0) are Nash equilibria (as well as any mixed strategy for player 1), but they aren't equally compelling because row 1 weakly dominates row 2 ∴ we could argue (6, 3) is the "best" Nash equilibrium
- **Why Best** - another argument for (6, 3) is that player 1 is only indifferent between A and B if player 2 plays X with probability 1; if player 2 plays Y (presumably by mistake), then row 1 is a better choice because it has a better payoff than row 2 in the case of player 2's mistake... this is trembling hand perfect argument

		Player 2	
		X	Y
Player 1	A	6, 3	5, 1
	B	6, 0	4, -1

- **Order Matters** - order of eliminating weakly dominated strategies matters; could cause problems if it changes the remaining set of equilibria (if so, we need to look closer at the structure of the game to see the order of moves, etc.); this is why some theorists argue against iterated weakly dominated strategies
- **How Likely** - some may argue that probability of having two payoffs being exactly equal is unlikely... until you consider extensive form where only difference between two strategies is irrelevant (i.e., have same payoff); e.g., strategy leading down equilibrium paths, 1 says X and 1 says Y



Pareto Superior - sum of payoffs being greater implies a better equilibrium

- **Example** - only 2 pure strategy Nash equilibria; which is better? most would argue (10, 10), but notice that A is weakly dominated by B so some theorists would say (3, 4) is more reasonable

		Player 2		
		X	Y	Z
Player 1	A	10, 10	4, 8	1, 2
	B	10, 5	5, 3	2, 6
	C	9, 1	7, 2	3, 4

B weakly dominates A

Trembling Hand Perfect - consider possibility that players make random errors; look at limit of probabilities of errors to get subset of original Nash equilibria

- **Theorem** - finite, 2 player game, trembling hand perfect and eliminating weakly dominated strategies are the same
- **Nonrandom Errors** - depend on strategy you're looking at (e.g., verbal directions ("B" sounds

(A,X) is Nash Eq. Assign prob of error to Y and Z

		Player 2		
		X	Y	Z
Player 1	A	10, 10	9, 8	7, 2
	B	10, 5	10, 3	8, 6

B weakly dominates A

With error E(A) < E(B) so player 1 should pick strategy B like "P"), typo (characters near each other on keyboard))

Random Errors - random noise with same probability distribution for all non-equilibrium strategies

Incorporate Into Game - create new payoff matrix based on probability 1 - ε of hitting strategy A and probability ε of making an error; enter expected values in each box (e.g., for (A,X) payoff would be (1 - ε)(1 - δ)(10) + ... (messy expression); each set of strategies will produce an expected payoff for each player: $E(\varepsilon, \delta)$; now goal is to find

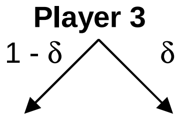
$$\lim_{\varepsilon,\delta \to 0} E(\varepsilon, \delta)$$

If errors may be significant, we may want something other than this limit, but need more info to model them (e.g., if one strategy is particularly bad so player is extra cautious, we could set up probability distribution over errors to make that strategy less likely)

Equilibrium Correspondence - closed set that relates $E(\varepsilon, \delta)$ to different values of ε and δ

Example - football coaches consider errors when calling plays (e.g., if you punt it may be blocked [your error] or it could be fumbled [opponent's error]); even if perfectly executed play may be best call, when accounting for errors, it may not be the best play to pick

Example - three player game with no weakly dominated strategies; assume (A,X,Left), (B,X,Left), and (B,Y,Right) are Nash equilibria and all players are aiming at (A,X,Left)



Player 2 (Player 1 at left)

		1 - ε	ε
		X	**Y**
1 - γ	**A**	$\underline{1}, \underline{1}, \underline{1}$	1, 0, 0
γ	**B**	$\underline{1}, \underline{1}, \underline{1}$	0, 0, ?

Player 2 (Player 1 at left)

		1 - ε	ε
		X	**Y**
1 - γ	**A**	0, 0, 0	0, 1, 1
γ	**B**	0, ?, 0	$\underline{1}, \underline{1}, \underline{1}$

Player 1 has A weakly dominating B in Left, but B weakly dominating A in Right

Look at player 1's expected value of A and B to determine which strategy is trembling hand perfect

$EV^A = 1(1 - \varepsilon)(1 - \delta) + 1(\varepsilon)(1 - \delta) + 0(1 - \varepsilon)(\delta) + 0(\varepsilon)(\delta) = 1 - \varepsilon - \delta + \varepsilon\delta + \varepsilon - \varepsilon\delta = 1 - \delta$

$EV^B = 1(1 - \varepsilon)(1 - \delta) + 0(\varepsilon)(1 - \delta) + 0(1 - \varepsilon)(\delta) + 1(\varepsilon)(\delta) = 1 - \varepsilon - \delta + 2\varepsilon\delta$

$EV^A - EV^B = \varepsilon - 2\varepsilon\delta = \varepsilon(1 - 2\delta)$

For small δ, $EV^A - EV^B > 0 \therefore$ A is better than B even though it doesn't dominate (not even weakly dominate); the only time player 1 is better using B is if both opponents have errors (less likely than one of the two making an error)

Conclusion - since player 1 will play A, we can remove the other two Nash equilibria for trembling hand perfect (only (A,X,Left) remains); THP throws out weakly dominated strategies and more

Own Errors -

Aim at A: $EV^A(1 - \gamma) + EV^B(\gamma)$

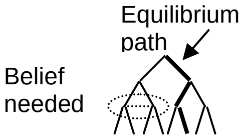
Aim at B: $EV^B(1 - \gamma) + EV^A(\gamma)$

Difference: $(1 - 2\gamma)(EV^A - EV^B)$; we know second term is > 0 so as long as $\gamma < 1/2$, player 1's choice is independent of γ

Realistic? - not really; football teams look at probability of their own errors (interceptions, fumble, etc.)

Extensive Form

- **Tree Rules** - a little more in depth on the details behind extensive form
  - **Successor** - nodes that can be reached from a given node by following arrows
  - **Immediate Successor** - node that is at the end of any arrow leading away from a given node
  - **Predecessor** - analogous to successors except we trace backward through the tree; also applies to **immediate predecessor**
  - **Path** - sequence of nodes that (1) starts with the initial node, (2) ends with a terminal node, and (3) has the property that successive nodes in the sequence are immediate successors of each other
  - **Rule 1** - every node is a successor of the initial node, and the initial node is the only one with this property
  - **Rule 2** - each node except the initial node has exactly one immediate predecessor; the initial node has no predecessors
  - **Rule 3** - multiple branches extending from the same node have different action labels
  - **Rule 4** - each information set contains decision nodes for only one of the players; if this weren't the case, at some point in the game, the players won't know who is to make a decision
  - **Rule 5** - all nodes in a given information set must have the same number of immediate successors and they must have the same set of action labels on the branches leading to these successors; if this weren't the case, the player would be able to distinguish between the nodes
- **Equilibrium Path** - what we observe if we watch the game
- **Refinements** - mostly target nodes that aren't on the equilibrium path because they don't affect expected payoffs in game
- **Perfect Information** - players know everything that happened before (i.e., aware of previous decisions by other players); equivalent to saying all information sets have only 1 node or saying it's a sequential game; beliefs are automatic (1 or 0 because player knows opponents' actions)



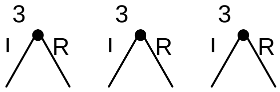
- **Imperfect Information** - information set has 2 or more nodes; in these cases, player must have belief about opponent's actions (i.e., assign a probability distribution)
  - **Refinement** - in general the probability distribution assigned doesn't matter if the information set is not on the equilibrium path, but refinement says people shouldn't believe someone will do something that's not rational
- **Sequential Rationality** - players ought to demonstrate rationality whenever they are called on to make decisions; optimal strategy for a player should maximize his or her expected payoff, conditional on every information set at which this player has the move
  - **Common Knowledge** - if sequential rationality is common knowledge, then each player will "look ahead" to consider what players will do in the future in response to his move at a particular information set
  - **Conditionally Dominated** - strategy s_i for player i is conditionally dominated if, contingent on reaching some information set of player i, there is another strategy σ_i that strictly dominates it
  - **Backward Induction** - simple version of finding conditionally dominated strategies; process of analyzing game from back to front; at each information set, remove non-optimal actions so set essentially becomes a terminal node; works best with perfect information because can't deal with information sets with multiple nodes; essentially same as eliminating weakly dominated strategies

Result - every finite game with perfect information has a pure-strategy Nash equilibrium; backward induction identifies an equilibrium

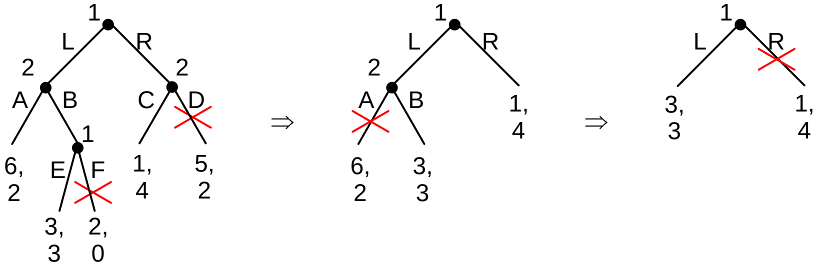
Good - generally yields unique equilibrium (most other refinements don't)

Bad - only works with finite time horizon

Example - strategies for player 3: (L,L,L), (R,L,L), etc. get identical payoffs with (*,L,L) so only difference comes from L or R in first component ∴ if one of these is better than the other, it weakly dominates (only 1 payoff is different)



Example - using backward indiction, we eliminate strategy F for player 1 and strategy D for player 2; move to next iteration; now eliminate strategy A for player 2; eliminate strategy R for player 1; Nash equilibrium therefore is (LE, BC), that is, strategies L and E for player 1 and B and C for player 2

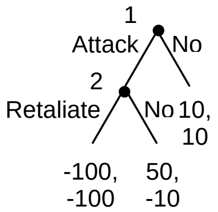


Subgame Perfect - if game comes to initial node of a subgame, the game will progress as if it were the subgame so we should find Nash equilibria to all subgames (don't believe people will do irrational things); closely related to throwing out weakly dominated strategies

Subgame - can't break up an information set and has to start from a single node;

Note: in game of perfect information, every node initiates a subgame

Example - in this game, it is not credible for player 2 to say he will retaliate; look at subgame starting with player 2's decision, Nash equilibrium is (Attack, No); **Note:** (No, Retaliate) is not a plausible equilibrium because it ignores the real-time dimension of the game



Player 2

		Retaliate	**No**
Player 1	**Attack**	-100, -100	$\underline{50}$, $\underline{-10}$
	No	$\underline{10}$, $\underline{10}$	10, $\underline{10}$

Multiple Nodes - still assume optimizing; first look for dominated strategies, then take limit of beliefs (sequence of believes as errors → 0; vs. trembling hand perfect which is sequence of equilibria as errors → 0)

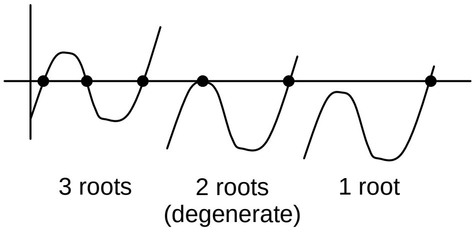
Example - **11.3**. This exercise explores how, in a mixed-strategy equilibrium, players must put positive probability only on best responses. Consider the game in the following figure.

Player 2

			q_1	q_2	q_3
			L	**M**	**R**
Player 1	p_1	**U**	$\underline{x}$, $\underline{x}$	x, 0	x, 0
	p_2	**C**	0, x	2, 0	0, 2
	p_3	**D**	0, x	0, 2	2, 0

Compute the pure and mixed Nash equilibria for this game and note how they depend on x. In particular, what is the difference between $x > 1$ and $x < 1$?

- **Note:** p_1, p_2, p_3 & q_1, q_2, q_3 are probabilities each player assigns to a strategy when playing a mixed strategy; $p_3 = 1 - p_2 - p_2$ and $q_3 = 1 - q_1 - q_2$ (only use p_3 and q_3 to make notation look simpler)
- (U, L) is only pure strategy Nash equilibrium (as long as $x > 0$)
- **Mixed strategies** - have to look at all combinations
  - **Fully Mixed** - using all three strategies implies $EV_U = EV_C = EV_D$ and $EV_L = EV_M = EV_R$
    - $EV_U = xq_1 + xq_2 + xq_3 = x$
    - $EV_C = 2q_2$
    - $EV_D = 2q_3 = 2(1 - q_1 - q_2)$
    - $x = 2q_2 \Rightarrow q_2 = x/2$
    - $2(1 - q_1 - q_2) = x \Rightarrow (1 - q_1 - x/2) = x/2 \Rightarrow q_1 = 1 - x$
    - Note: this strategy requires $x < 1$ (in order to have $q_1 < 1$); can have $x = 1$, but that results in degenerate solution (one of the three strategies will have probability 0 so it'll be same as paired strategy case)
    - $EV_L = xp_1 + xp_2 + xp_3 = x$
    - $EV_M = 2p_3 = 2(1 - p_1 - p_2)$
    - $EV_R = 2p_2$... these are basically the same as player 1's
    - Fully mixed strategy: $p_1 = q_1 = 1 - x$; $p_2 = p_3 = q_2 = q_3 = x/2$
  - **Paired Strategies** - using strategies two at a time has 3 cases for each player; looking at all combinations means 3x3 = 9 possibilities; first check for dominated strategies so we do least amount of work possible.
    - (U,C) and (L,M) (i.e., $p_3 = q_3 = 0$) $\Rightarrow$ L dominates M ($\therefore$ not a Nash equilibrium)
    - (U,C) and (L,R) (i.e., $p_3 = q_2 = 0$) $\Rightarrow$ U dominates C
    - (U,C) and (M,R) (i.e., $p_3 = q_1 = 0$) $\Rightarrow$ R weakly dominates M
    - (U,D) and (L,M) (i.e., $p_2 = q_3 = 0$) $\Rightarrow$ U dominates D
    - (U,D) and (L,R) (i.e., $p_2 = q_2 = 0$) $\Rightarrow$ L dominates R
    - (U,D) and (M,R) (i.e., $p_2 = q_1 = 0$) $\Rightarrow$ M weakly dominates R
    - (C,D) and (L,M) (i.e., $p_1 = q_3 = 0$) $\Rightarrow$ L dominates M
    - (C,D) and (L,R) (i.e., $p_1 = q_2 = 0$) $\Rightarrow$ L dominates R
    - (C,D) and (M,R) (i.e., $p_1 = q_1 = 0$) $\Rightarrow$ nothing dominates... can have a mixed strategy
      - $EV_C = 2q_2 = EV_D = 2q_3 = 2(1 - q_1 - q_2) \Rightarrow q_2 = q_2 = 1/2$
      - $EV_M = 2p_3 = 2(1 - p_1 - p_2) = EV_R = 2p_2 \Rightarrow p_2 = p_3 = 1/2$
      - This only looked at first rule (EV of all strategies being used are equal); now need to check second rule (EV of strategies being used $\geq EV$ of strategies not used); $EV_C = EV_D \geq EV_U \Rightarrow 2q_2 \geq x \Rightarrow x \leq 1$ (same result when looking at $EV_M = EV_R \geq EV_L$)
  - **Solution** -
    - $x > 1$ 1 equilibrium: (U,L) (pure strategy only)
    - $x = 1$ 2 equilibria: (U,L) (pure strategy) and $p_1 = q_2 = 0$ and $p_2 = p_3 = q_2 = q_3 = 1/2$
    - $x < 1$ 3 equilibria: previous 2 plus $p_1 = q_1 = 1 - x$; $p_2 = p_3 = q_2 = q_3 = x/2$
  - **Theorem** - generally will have odd number of equilibria; case of even number is generally degenerate... like polynomial:



Incomplete Information

Asymmetric Information - general term for players having different amounts of information
- **Private Information** - player knows something about his own actions that other players don't observe
- **Type of Information** - information could be regarding players' actions, payoffs, or some other intangible aspect of the game (e.g., weather forecast)

Moves of Nature - random events used to incorporate private information

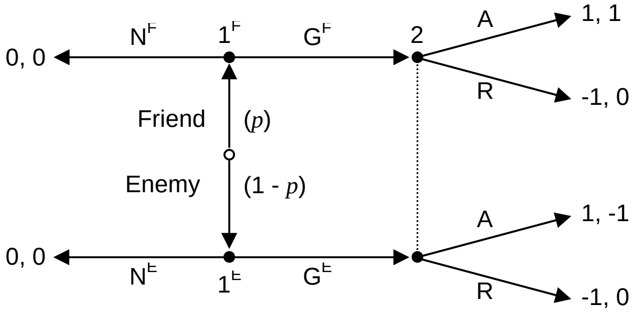
Incomplete Information - refers to games having moves of nature that generate asymmetric information between players; causes problems when trying to maximize (don't have all necessary information)
- **Harsanyi** - paper in Management Science (1967) on how to convert game with incomplete information to game of imperfect, but complete information by inserting random move by nature
  - **Imperfect** - information sets with more than one node (player doesn't know what opponent did or has simultaneous move)
  - **Complete** - everyone knows structure of game
  - **Nature** - "decisions" made according to a fixed probability distribution; no payoff numbers are associated with nature... "decisions" are referred to as **chance nodes** (depicted with open circles)
- **Type** - term used to indicate different moves of nature that a single player privately observes (e.g., different types of home buyers distinguished by different valuations of the house)
  - **Rationality** - requires player who knows his own type to think about what he would have done had he boon another type (i.e., determine best reply for each type)
  - **Bayesian Normal Form** - translates extensive form game with incomplete information into normal form with strategy for each player type and payoffs computed by averaging over random events in game (e.g., if 1 player has 2 types and 2 strategies for each type, there will be $2^2 = 4$ strategies; n types and m strategies means m^n strategies

2 Ways to Solve - each way yields same solution; pick whichever seems easier
- **Incomplete Information** - find best reply for each player type; end up with $(m - 1)n$ equations and $(m - 1)n$ unknowns; unknowns being probability of taking certain strategy given specific player type; note n player types and m strategies so if there are 3 types each with 2 strategies, there are 3 equations (for each player)... see poker example; this technique is not very intuitive and solving the equations is difficult, but could be easier (or less time consuming) than other technique
- **Complete, Imperfect Information** - use m^n strategies in extensive form or Bayesian normal form... ends up with huge matrix; time consuming (but methodical) to enter all the payoffs; after that, it's pretty easy to find pure strategy Nash equilibria (if any exist); finding mixed strategies is much more difficult... not recommended if more than 4x4

Gift Example (book Chpt 24)

Nature first determines the type of player 1 (Friend with probability p or Enemy with probability $1 - p$). Player 1 observes nature's move (knows own type), but player 2 doesn't. Player 1 decides whether to give a gift to player 2, then player 2 decides whether to accept it or not.
- **Bayesian Normal Form** - player 1 has 2 strategies (give & not) and 2 types $\therefore$ 4 strategies total; player 2 has 2 strategies

Payoffs - take expected payoffs based on player 1's type (e.g., for G^FG^E A cell in table: player 1 gets 1 regardless of his type; player 2 gets 1 if player 1 is friend and -1 if enemy ∴ payoff = $1(p) + -1(1 - p) = 2p - 1$



		Player 2	
		A	R
Player 1	G^FG^E	1, 2p - 1	-1, 0
	G^FN^E	p, p	-p, 0
	N^FG^E	1 - p, p - 1	p - 1, 0
	N^FN^E	0, 0	0, 0

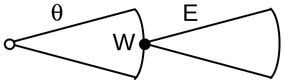
Principal-Agent Problem (book Chpt 25)

Principal-Agent - refers to situation in which one party (the principal) hires another party (the agent) to work on a project on his behalf

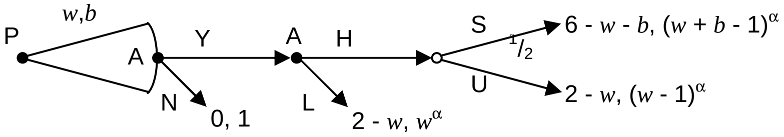
Moral Hazard - stands for setting in which the agent's effort is not verifiable, so the parties cannot write an externally enforced contract specifying a payment as a function of effort

$q(E,\theta)$ - output is function of effort and nature, neither is observed by the principal (owner/manager), but agent (worker) observes θ before deciding E; want to sent up incentives based on E, but also account for θ

Simple Version -

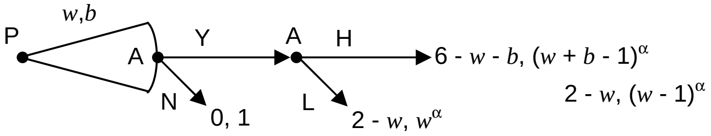


Book Version - Pat is owner and wants to hire Allen; offers contract specifying a wage (w) and bonus (b); Allen then decides whether to accept the contract or not (payoff of 1 through other employment); then Allen decides whether to expend high effort (cost 1) or low effort (no cost); if Allen does high effort, there's a 50-50 chance of the project being successful (but no chance if he does low effort); successful project is worth 6 to Pat and b to Allen; unsuccessful project is worth 2 to Pat and nothing to Allen (but he gets w either way); Allen's utility function is $u_A(x) = x^\alpha$; Pat is risk neutral (usual assumption for principal)



Use backward induction; first get expected payoffs for project

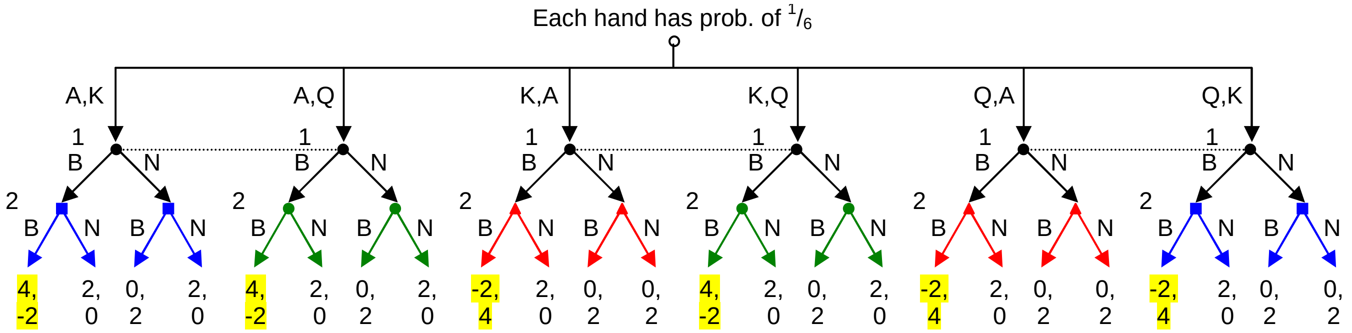
Note: 1/2(6 - w - b) + 1/2(2 - w) = 3 - w/2 - b/2 + 1 - w/2 = 4 - 2 - b/2



Poker Example (from class)

3 card deck: A, K, Q; each player gets 1 card which he sees, but opponents don't see each others' card; each pays $1 in pot prior to deal; after they see their cards, players simultaneously decide to bet ($2) or not bet; if both bet or neither bets, player with higher card takes money in pot (with bets, that's $4 net; $2 without bets); if only one player bets, he wins the pot automatically ($2; opponent gets $0)

Complete, Imperfect Information - can use extensive form (big, but easy) or Bayesian normal form; end up with 3 types of players (A, K, Q) each with 2 strategies (B, N) for a total of 2^3 = 8 strategies for each player; you can see this in the extensive form because each player has 3 information sets (must make a decision in each of them); in each information set, the player only knows his card ∴ form information sets by joining all decision nodes where player knows his card is the same... easy to mark for player 1, but ugly for player 2 (used colors & shapes: red triangles for A, blue squares for K, green circles for Q)



Bayesian Normal Form - have to find expected payoffs for each combination of strategies... 8x8 means 64 of them! (this isn't easy)

BBB and BBB - both players always bet (highlighted payoffs)

Player 1 - 1/6(4 + 4 + -2 + 4 + -2 + -2) = 1

Player 2 - 1/6(-2 + -2 + 4 + -2 + 4 + 4) = 1

Symmetric - payoff in cell ij = reverse payoffs in cell ji

Constant Sum - sum of payoffs = 2 (from $2 in pot; after that, 1 player wins an additional $2 only if the other one loses $2)

Solution - only 1 pure strategy Nash equilibrium (BNN, BNN); that means both players always bet with A, but never bet with K or Q; could be mixed strategies, but they'd be very hard to find this way (have to try all combos of 2, 3, etc.)... try imperfect info technique

Player 2

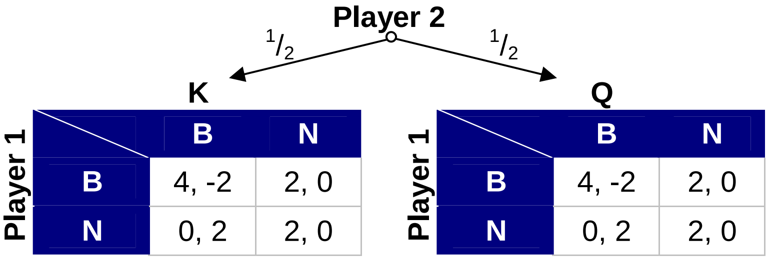
		BBB	BBN	BNB	BNN	NBB	NBN	NNB	NNN
Player 1	**BBB**	1, 1	$^{1}/_{3}$, $\underline{^{5}/_{3}}$	$\underline{^{4}/_{3}}$, $^{2}/_{3}$	$^{2}/_{3}$, $^{4}/_{3}$	$\underline{^{7}/_{3}}$, $-^{1}/_{3}$	$\underline{^{5}/_{3}}$, $^{1}/_{3}$	$\underline{^{8}/_{3}}$, $-^{2}/_{3}$	$\underline{2}$, 0
	BBN	$\underline{^{5}/_{3}}$, $^{1}/_{3}$	1, 1	$\underline{^{4}/_{3}}$, $^{2}/_{3}$	$^{2}/_{3}$, $\underline{^{4}/_{3}}$	$\underline{^{7}/_{3}}$, $-^{1}/_{3}$	$\underline{^{5}/_{3}}$, $^{1}/_{3}$	2, 0	$^{4}/_{3}$, $^{2}/_{3}$
	BNB	$^{2}/_{3}$, $\underline{^{4}/_{3}}$	$^{2}/_{3}$, $\underline{^{4}/_{3}}$	1, 1	$\underline{1}$, 1	$^{4}/_{3}$, $^{2}/_{3}$	$^{4}/_{3}$, $^{2}/_{3}$	$^{5}/_{3}$, $^{1}/_{3}$	$^{5}/_{3}$, $^{1}/_{3}$
	BNN	$^{4}/_{3}$, $^{2}/_{3}$	$\underline{^{4}/_{3}}$, $^{2}/_{4}$	1, $\underline{1}$	$\underline{1}$, $\underline{1}$	$^{4}/_{3}$, $^{2}/_{3}$	$^{4}/_{3}$, $^{2}/_{3}$	1, 1	1, 1
	NBB	$-^{1}/_{3}$, $\underline{^{7}/_{3}}$	$-^{1}/_{3}$, $\underline{^{7}/_{3}}$	$^{2}/_{3}$, $^{4}/_{2}$	$^{2}/_{3}$, $^{4}/_{3}$	1, 1	1, 1	2, 0	$\underline{2}$, 0
	NBN	$^{1}/_{3}$, $\underline{^{5}/_{3}}$	$^{1}/_{3}$, $\underline{^{5}/_{3}}$	$^{2}/_{3}$, $^{4}/_{3}$	$^{2}/_{3}$, $^{4}/_{3}$	1, 1	1, 1	$^{4}/_{3}$, $^{2}/_{3}$	$^{4}/_{3}$, $^{2}/_{3}$
	NNB	$-^{2}/_{3}$, $\underline{^{8}/_{3}}$	0, 2	$^{1}/_{3}$, $^{5}/_{3}$	$\underline{1}$, 1	0, 2	$^{2}/_{3}$, $^{4}/_{3}$	1, 1	$^{5}/_{3}$, $^{1}/_{3}$
	NNN	0, $\underline{2}$	$^{2}/_{3}$, $^{4}/_{3}$	$^{1}/_{3}$, $^{5}/_{3}$	$\underline{1}$, 1	0, $\underline{2}$	$^{2}/_{3}$, $^{4}/_{3}$	$^{1}/_{3}$, $^{5}/_{3}$	1, 1

Imperfect Information - consider the card player 1 has and look at the probability of betting in each case

Comments -

1. Payoff tables are same when player 1 has A (because it beats other 2 cards); same happens when player 1 has Q; still need to worry about both payoff tables because we need to worry about opponent's strategy
2. Note for player 1 with an A, B weakly dominates N... since we're interested in finding all Nash equilibria, we can't eliminate weakly dominated strategies because we may eliminate a Nash equilibrium
3. Player 1 knowing his type influences his belief on player 2's type (i.e., 50-50 of having one of remaining 2 cards); that's not always the case
4. Normally we'd repeat this process for player 2, but in this case it's completely symmetrical so we don't have to
5. Symmetry in the game means there will be a symmetric equilibria, but there could be others so we have to do this the long way; we do get to take advantage of symmetry for the computations

Player 1 with an A



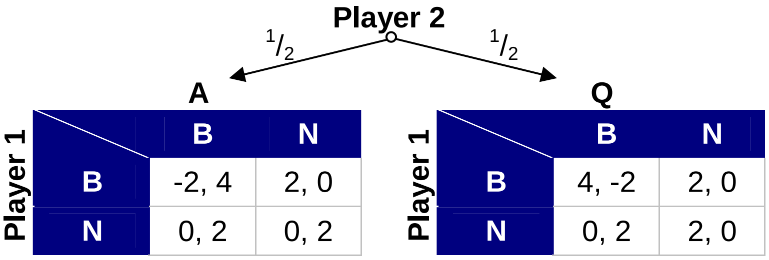
K:

	B	N
B	4, -2	2, 0
N	0, 2	2, 0

Q:

	B	N
B	4, -2	2, 0
N	0, 2	2, 0

Player 1 with a K



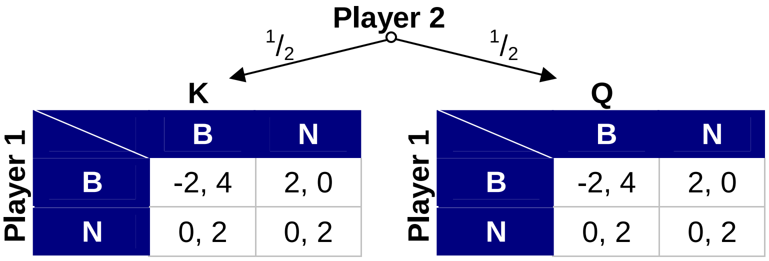
A:

	B	N
B	-2, 4	2, 0
N	0, 2	0, 2

Q:

	B	N
B	4, -2	2, 0
N	0, 2	2, 0

Player 1 with an Q



K:

	B	N
B	-2, 4	2, 0
N	0, 2	0, 2

Q:

	B	N
B	-2, 4	2, 0
N	0, 2	0, 2

Solving - find best reply function for each player; looking at either choosing B or N (or mixed strategy) based on whether EV^B >, <, or = to EV^N; using π_i and θ_i to represent the probability that player 1 and player 2 (respectively) will bet given they have card i (i = A,K,Q)

Player 1:

$$\pi_A = \begin{cases} 1 & > 0 \\ [0,1] \text{ as } EV^B - EV^N & = 0 \\ 0 & < 0\text{, where} \end{cases}$$

$EV^B = {}^1/_2(4\theta_K + 2(1 - \theta_K) + 4\theta_Q + 2(1 - \theta_Q)) = \theta_K + \theta_Q + 2$

$EV^N = {}^1/_2(0\theta_K + 2(1 - \theta_K) + 0\theta_Q + 2(1 - \theta_Q)) = 2 - \theta_K - \theta_Q$

$\therefore EV^B - EV^N = \theta_K + \theta_Q$

Repeat for player 1 with a K

$EV^B = {}^1/_2(-2\theta_A + 2(1 - \theta_A) + 4\theta_Q + 2(1 - \theta_Q)) = -2\theta_A + \theta_Q + 2$

$EV^N = {}^1/_2(0\theta_A + 0(1 - \theta_A) + 0\theta_Q + 2(1 - \theta_Q)) = 1 - \theta_Q$

$\therefore EV^B - EV^N = -2\theta_A + 2\theta_Q + 1$

Repeat for player 1 with a Q

$EV^B = {}^1/_2(-2\theta_A + 2(1 - \theta_A) + -2\theta_K + 2(1 - \theta_K)) = -2\theta_A -2\theta_K + 2$

$EV^N = {}^1/_2(0\theta_A + 0(1 - \theta_A) + 0\theta_K + 0(1 - \theta_K)) = 0$

$\therefore EV^B - EV^N = -2\theta_A - 2\theta_K + 2$

Player 2:

Because of symmetry in the game, player 2's probabilities (θ_A, θ_K, θ_Q) will be exactly the same as the corresponding probabilities for player 1 (i.e., swap θ for π and vice versa for all of player 1's formulas)

Summary: (6 equations and 6 unknowns)

$$\pi_A = \begin{cases} 1 & > 0 \\ [0,1] \text{ as } \theta_K + \theta_Q & = 0 \\ 0 & < 0 \end{cases} \quad (1) \qquad \theta_A = \begin{cases} 1 & > 0 \\ [0,1] \text{ as } \pi_K + \pi_Q & = 0 \\ 0 & < 0 \end{cases} \quad (4)$$

$$\pi_K = \begin{cases} 1 & > \theta_A - \theta_Q \\ [0,1] \text{ as } {}^1/_2 & = \theta_A - \theta_Q \\ 0 & < \theta_A - \theta_Q \end{cases} \quad (2) \qquad \theta_K = \begin{cases} 1 & > \pi_A - \pi_Q \\ [0,1] \text{ as } {}^1/_2 & = \pi_A - \pi_Q \\ 0 & < \pi_A - \pi_Q \end{cases} \quad (5)$$

$$\pi_Q = \begin{cases} 1 & > \theta_A + \theta_K \\ [0,1] \text{ as } 1 & = \theta_A + \theta_K \\ 0 & < \theta_A + \theta_K \end{cases} \quad (3) \qquad \theta_Q = \begin{cases} 1 & > \pi_A + \pi_K \\ [0,1] \text{ as } 1 & = \pi_A + \pi_K \\ 0 & < \pi_A + \pi_K \end{cases} \quad (6)$$

Solving this system requires some creativity to be easy, but it can also be solved if you're very methodical. The basic idea is to start with one of the three possible values for one of the unknowns and then either find values for all the other unknowns that give a solution or find a contradiction (e.g., assume something = 1 then show that it's not). In this case, intuition would suggest always betting with the A... note that you'll never not bet with it (i.e., π_A or $\theta_A = 0$) because then (1) implies $\theta_K + \theta_Q < 0$ and (4) implies $\pi_K + \pi_Q < 0$ which is not possible with probabilities. In order to the theory of always betting with an A, let's assume $\pi_A < 1$ (Note: a more correct notation would be $\pi_A \in (0,1)$)

Assume $\pi_A < 1$

(1) implies $\theta_K + \theta_Q = 0 \Rightarrow \theta_K = \theta_Q = 0$ (because probabilities can't be negative)

Now (5) implies ${}^1/_2 < \pi_A - \pi_Q$ and (6) implies $1 < \pi_A + \pi_K$

From the first inequality we can rule out $\pi_Q = 1$, but that's about it; we're stuck here and have to make another assumption

Because of the symmetry, we'll assume $\theta_A < 1$

(4) implies $\pi_K + \pi_Q = 0 \Rightarrow \pi_K = \pi_Q = 0$

But now look at the implication from (6) above: $1 < \pi_A + \pi_K \Rightarrow$ (sub $\pi_K = 0$) $1 < \pi_A$

That violates our initial assumption that $\pi_A < 1$

$\therefore$ we now know that $\pi_A = 1$ (and by symmetry $\theta_A = 1$)

(1) implies $\theta_K + \theta_Q > 0$ and (2) implies $\pi_K + \pi_Q > 0$

We're stuck again so we need another assumption; look at $\pi_K = 1$

Assume $\pi_K = 1$

(2) implies ${}^1/_2 > \theta_A - \theta_Q = 1 - \theta_Q \Rightarrow \theta_Q > {}^1/_2$

But now (6) implies $1 = \pi_A + \pi_K \Rightarrow 1 = 1 + \pi_K \Rightarrow \pi_K = 0$ which violates our assumption

Assume $\pi_K \in [0,1]$

(2) implies ${}^1/_2 = \theta_A - \theta_Q = 1 - \theta_Q \Rightarrow \theta_Q = {}^1/_2$

But now (6) implies $1 = \pi_A + \pi_K \Rightarrow 1 = 1 + \pi_K \Rightarrow \pi_K = 0$ which violates our assumption

$\therefore$we now know that $\pi_K = 0$ (and by symmetry $\theta_K = 0$)... if there is a solution

(2) implies ${}^1/_2 < \theta_A - \theta_Q \Rightarrow {}^1/_2 < 1 - \theta_Q \Rightarrow \theta_Q < {}^1/_2$

(6) implies $1 = \pi_A + \pi_K$ which is correct because $\pi_A = 1$ and $\pi_K = 0$

Similarly (using (5) and (3)) we get $\pi_Q < {}^1/_2$

Final solution is:

$\pi_A = \theta_A = 1$
$\pi_K = \theta_K = 0$
$\pi_Q \in [0,^1/_2]$ and $\theta_Q \in [0,^1/_2]$
That is, both players always bet with an A and never bet with a K; if they have a Q, the can either not bet (that's the pure strategy Nash equilibrium we found earlier) or they can play a mixed strategy were the probability of betting with a Q is anywhere between 0 and $^1/_2$ (**Note:** this is bluffing because the Q can't beat either of the other cards; if the players don't bet simultaneously, a player with a Q may be in the hopes that the opponent with a K thinks he has an A and doesn't bet)

Want More?

Can repeat this game with only 2 cards to make it simpler, say player 1 either gets A or Q and player 2 either gets K or J (all with probability $^1/_2$) I won't go into all the math, but these some of the steps leading up to the results:

Complete, Imperfect Information - drawing extensive form is simpler than previous version; hardest part is getting the payoffs for the Bayesian strategic form

Player 2

		BB	BN	NB	NN
Player 1	**BB**	$\underline{^5/_2}, -^1/_2$	$^3/_2, \underline{^1/_2}$	$\underline{3}, -1$	$\underline{2}, 0$
	BN	2, 0	$\underline{2}, 0$	$^3/_2, \underline{^1/_2}$	$^3/_2, \underline{^1/_2}$
	NB	$^1/_2, \underline{^3/_2}$	$^1/_2, \underline{^3/_2}$	2, 0	$\underline{2}, 0$
	NN	$0, \underline{2}$	1, 1	$^1/_2, ^3/_2$	$^3/_2, ^1/_2$

Results -

- There's no pure strategy Nash equilibrium
- NN is strictly dominated by BB for player 1
- We can now rule out weakly dominated strategies (only concerned with mixed strategies)
- BB weakly dominated by BN and NB weakly dominated by NN for player 2
- Now BB weakly dominates NB for player 1
- Remaining 2x2 game has no pure strategy equilibrium and is symmetric so figure mixed strategy equilibrium is 50-50 for both players
- End result is players mixing of BB and BN for player 1 (i.e. always bet with A, and mix with Q) and BN and NN for player 2 (i.e., never bet with J and mix with K)
- Finding this solution is shown after the incomplete information solution

Incomplete Information - set up payoff tables from each type of player like we did before... note that it's not symmetric anymore; should end up with this:

$$\pi_A = \begin{cases} 1 & > 0 \\ [0,1] \text{ as } \theta_K + \theta_J & = 0 \\ 0 & < 0 \end{cases} \quad (1) \qquad \theta_K = \begin{cases} 1 & > \pi_A - \pi_Q \\ [0,1] \text{ as } ^1/_2 & = \pi_A - \pi_Q \\ 0 & < \pi_A - \pi_Q \end{cases} \quad (3)$$

$$\pi_Q = \begin{cases} 1 & > \theta_K - \theta_J \\ [0,1] \text{ as } ^1/_2 & = \theta_K - \theta_J \\ 0 & < \theta_K - \theta_J \end{cases} \quad (2) \qquad \theta_J = \begin{cases} 1 & > \pi_A + \pi_Q \\ [0,1] \text{ as } 1 & = \pi_A + \pi_Q \\ 0 & < \pi_A + \pi_Q \end{cases} \quad (4)$$

Result -

$\pi_A = 1$, $\pi_Q = ^1/_2$, $\theta_K = ^1/_2$, $\theta_J = 0$

Player 2

			θ	$1 - \theta$
			BN	**NN**
Player 1	π	**BB**	$^{3}/_{2}$, $\underline{^{1}/_{2}}$	$\underline{2}$, 0
	$1 - \pi$	**BN**	$\underline{2}$, 0	$^{3}/_{2}$, $\underline{^{1}/_{2}}$

Good Review for Mixed Strategies

This payoff matrix comes from removing the strictly and weakly dominated strategy in the "simple" (2 card) poker game; technically, we still need to worry about the whole thing, but first we're going to solve for π and θ (note that these are the same as π_Q and θ_K in the poker game... look at the strategies and it should make sense)

1. If two (or more) strategies are played with positive probability by a player, then the expected value of those strategies are equal to each other

   Player 1:

   $EV^{BB} = EV^{BN} \Rightarrow {}^{3}/_{2}\theta + 2(1 - \theta) = 2\theta + {}^{3}/_{2}(1 - \theta) \Rightarrow \theta = {}^{1}/_{2}$

   Player 2:

   $EV^{BN} = EV^{NN} \Rightarrow {}^{1}/_{2}\pi + 0(1 - \pi) = 0\pi + {}^{1}/_{2}(1 - \pi) \Rightarrow \pi = {}^{1}/_{2}$

2. The expected payoff of strategies with positive probability must be at least as great as ($\geq$) the expected value from strategies with zero probability

   To check this we have to go back to the original 4x4 payoff matrix... basically we're looking for EV^{NB} and EV^{NN} for player 1 to be < EV^{BB} (or EV^{BN}) and EV^{BB} and EV^{NB} for player 2 to be < EV^{BN} (or EV^{NN})

   Player 1: probabilities for player 2 playing BB, BN, NB, and NN are 0, $^{1}/_{2}$, 0, $^{1}/_{2}$, respectively

   $EV^{BB} = {}^{5}/_{2}(0) + {}^{3}/_{2}({}^{1}/_{2}) + 3(0) + 2({}^{1}/_{2}) = {}^{7}/_{4}$

   $EV^{BN} = 2(0) + 2({}^{1}/_{2}) + {}^{3}/_{2}(0) + {}^{3}/_{2}({}^{1}/_{2}) = {}^{7}/_{4}$... as expected since EVBB = EVBN

   $EV^{NB} = {}^{1}/_{2}(0) + {}^{1}/_{2}({}^{1}/_{2}) + 2(0) + 2({}^{1}/_{2}) = {}^{5}/_{4} < {}^{7}/_{4}$

   $EV^{NN} = 0(0) + 1({}^{1}/_{2}) + {}^{1}/_{2}(0) + {}^{3}/_{2}({}^{1}/_{2}) = {}^{5}/_{4} < {}^{7}/_{4}$... works out correctly

   Player 2: probabilities for player 1 playing BB, BN, NB, and NN are $^{1}/_{2}$, $^{1}/_{2}$, 0, 0, respectively

   $EV^{BB} = -{}^{1}/_{2}({}^{1}/_{2}) + 0({}^{1}/_{2}) + {}^{3}/_{2}(0) + 2(0) = -{}^{1}/_{4}$

   $EV^{BN} = {}^{1}/_{2}({}^{1}/_{2}) + 0({}^{1}/_{2}) + {}^{3}/_{2}(0) + 1(0) = {}^{1}/_{4}$

   $EV^{NB} = -1({}^{1}/_{2}) + {}^{1}/_{2}({}^{1}/_{2}) + 0(0) + {}^{3}/_{2}(0) = -{}^{1}/_{4}$

   $EV^{NN} = 0({}^{1}/_{2}) + {}^{1}/_{2}({}^{1}/_{2}) + 0(0) + {}^{1}/_{2}(0) = {}^{1}/_{4}$... works out correctly

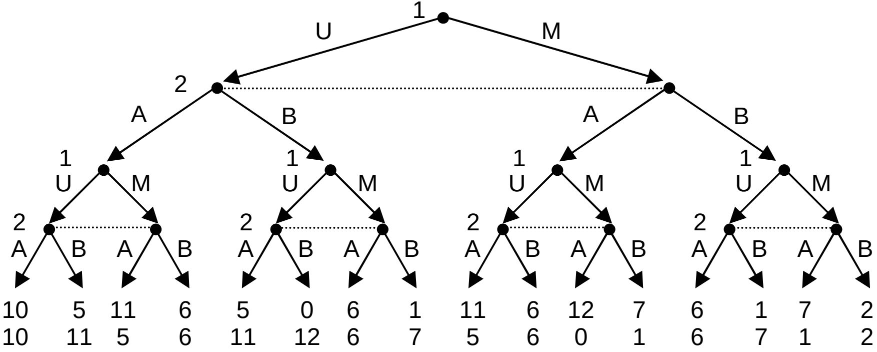
Repeated Play

		Player 2	
		A	B
Player 1	U	5, 5	0, 6
	M	6, 0	1, 1

Prisoners Dilemma - could coordinate to get to (A,U) (best payoff), but incentives point away from that so it's not rational to expect the coordination to work

Finitely Repeated -

- **2 Periods** - each period is simultaneous, but players know what happened in first period before playing the second; basically end up with 5 information sets for each player, each with 2 strategies... that's 25 strategies... probably don't want to do that in strategic form; we'll look at subgame perfection



- **Discounting** - note that if we discount payoffs in future periods, say δ for each period, we can't just add them up like this; for example, (UU,AA) would be $(5 + 5\delta, 5 + 5\delta)$; discounting is critical for infinitely repeated games (in order to have bounded limits of payoffs)
- Looking at all the second periods games, they're identical with Nash equilibria of (M,B) (payoff of (1,1)). Since they're all the same, we can ignore them and just look at the first period; in this case the Nash equilibrium is (M,B)... ∴ subgame perfect Nash equilibrium for this game is (MM,BB)
- **No Cooperation** - finite repetition doesn't solve the prisoners' dilemma (or any game with unique Nash equilibrium); that unique equilibrium will be the subgame perfect Nash equilibrium for the finitely repeated game
- **Penalties** - if we have another strategy that is very bad all the way around, but gives a pure strategy Nash equilibrium like the one shown here, we can get cooperation because final stage has to be either (B,M) or (D,C), but first stage can be anything as long as combination forms a Nash equilibrium

		Player 2		
		A	B	C
Player 1	U	5, 5	0, 6	-6, -6
	M	6, 0	1, 1	-6, -6
	D	-6, -6	-6, -6	-4, -4

  - **# Strategies** - in this case there are 10 information sets each with 3 strategies: 3^{10} = 59,049... that's a lot!
  - **Specific Choice** - since there are so many strategies, let's just consider the case of cooperating in first round, then either going for the (1,1) payoff if the opponent cooperated or the (-4,-4) payoff if he didn't
    - Strategy for Player 1: (U, if UA then M, else D)
    - Strategy for Player 2: (A, if UA then B, else C)
    - **Credible Threat** - each player is making a credible threat because it leads to a Nash equilibrium in the last round
    - **Nash?** - need to check strategy for each player to see if he has an incentive to change

- **Player 1** - changing to D in first round is obviously bad; changing to M gains 1 unit of payoff (based on player 2's strategy), but then player 1 gets -4 instead of 1 (loss of 5) in second round
  - **With Discounts** - if we were using discounts we'd have to compare the gain of 1 now with a loss of 5 in the second round which is worth -5δ; ∴ we'd only switch if $\delta < 1/5$ (i.e., discounting a lot; payoff now is much more important than payoff later); another way to look at this is to compare $(6 - 4\delta)$ to $(5 + \delta)$... get same result, only switch if $\delta < 1/5$
- **Player 2** - exact same logic since it's symmetric
- ∴ this specific choice we looked at is a subgame perfect Nash equilibrium
- **Note:** if the payoff for (D,C) goes more negative to (-6,-6) or worse, this strategy is no longer subgame perfect

Trigger Strategy - cooperate, but punish if opponent doesn't cooperate

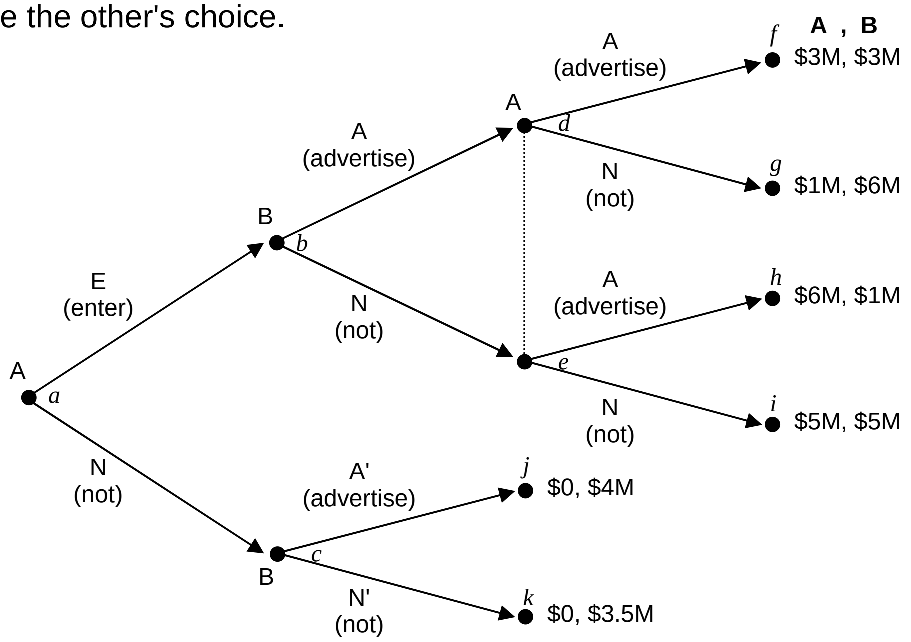
- **When to Penalize** - if there are 20 rounds, and penalty comes at the end, we're looking at a discount of δ^{19}... probably too small to make a difference; opposite extreme is grim trigger
- **Grim Trigger** - punish forever
- **Need "Bad" Equilibrium** - only way to punish in finite game is to have a "bad" equilibrium (because threat must use an equilibrium for strategy to be subgame perfect)

Infinitely Repeated - don't need a "bad" equilibrium to punish since we're not worried about subgame perfect (never get to the end); we do have to have discounted payoffs with $\delta < 1$ to allow payoffs (which are infinite sums) to converge

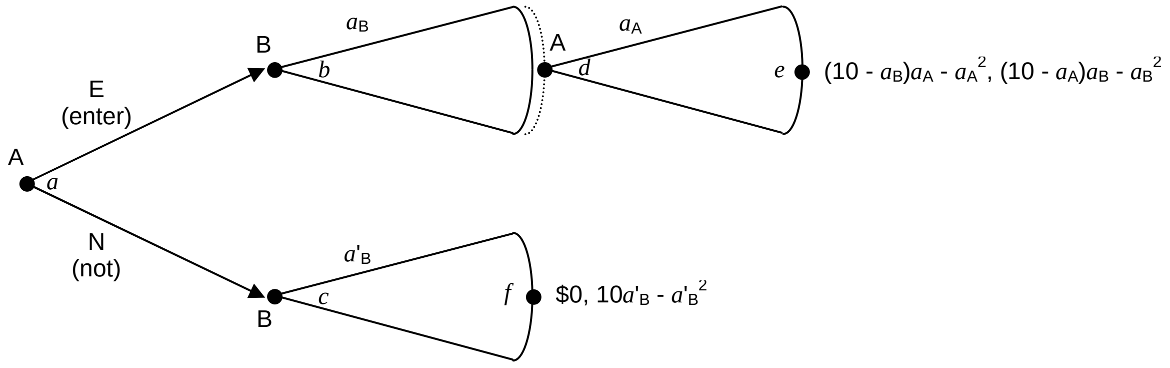
- **Grim Trigger Strategy** - here's an example for infinitely repeated game:
  - Player 1: Initial State: U; Every stage after: if previous stage was UA, then U, else M
  - Player 2: Initial State: A; Every stage after: if previous stage was UA, then A, else B
  - If player deviates in round 1, gain 1 (6 instead of 5), but lose 5 (-4 instead of 5) from period 2 to infinity (i.e., $-5(\delta + \delta^2 + \delta^3...) = -5(\delta/(1 - \delta))$)... again end up with cooperate if $\delta > 1/5$
- **Too Many Equilibria** - problem with infinitely repeated games is that there are lots of equilibria... all equally compelling so even thought people used infinitely repeated games to show possibility of good outcome (e.g., cooperation), there's also possibility of bad outcome; there's no justification for one over the other (**Folk Theorem**)

2.1. (a) Represent the following game in extensive form. Firm A decides whether to enter firm B's industry. Firm B observes this decision. If firm A enters, then the two firms simultaneously decide whether to advertise. Otherwise, firm B alone decides whether to advertise. With two firms in the market, the firms earn profits of $3 million each if they both advertise and $5 million if they both do not advertise. If only one firm advertises, then it earns $6 million and the other firm earns $1 million. When firm B is solely in the industry, it earns $4 million if it advertises and $3.5 million if it does not advertise. Firm A earns $0 if it does not enter. **(b)** Suppose that instead of deciding whether to advertise, the firms decide how much to spend on advertising. With both firms in the industry, firm i earns gross revenues of $(10 - a_j)a_i - a_i^2$, where a_i is firm i's advertising level and a_j is the other firm's advertising level. With firm B alone in the market and spending a_B on advertising, it obtains $10a_B - a_B^2$ in gross revenues. Represent this new game in the extensive form. Note that the advertising choices are now continuous variables, so you must use arcs to represent them (as in Figures 2.8 and 2.9). Remember to add a dotted line where you wish to represent that one player does not observe the other's choice.

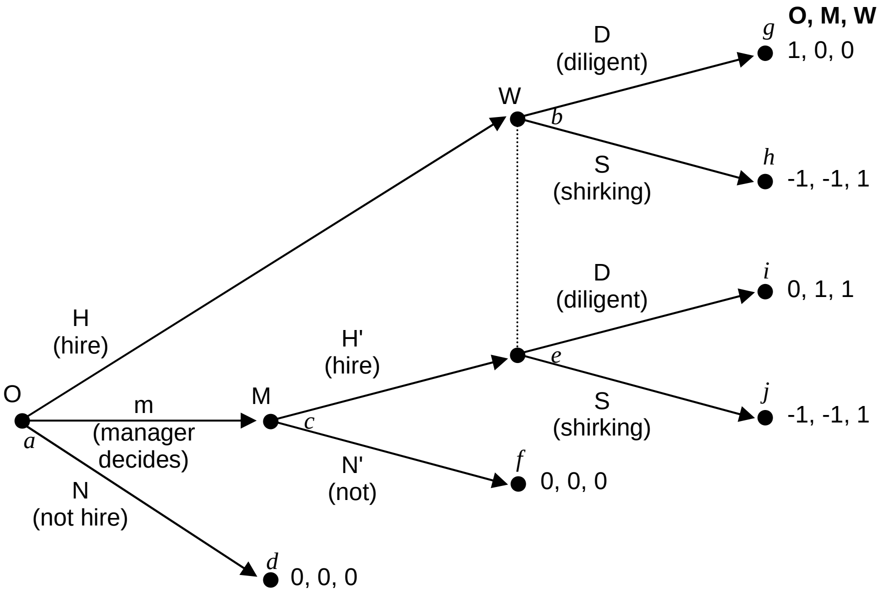
(a)



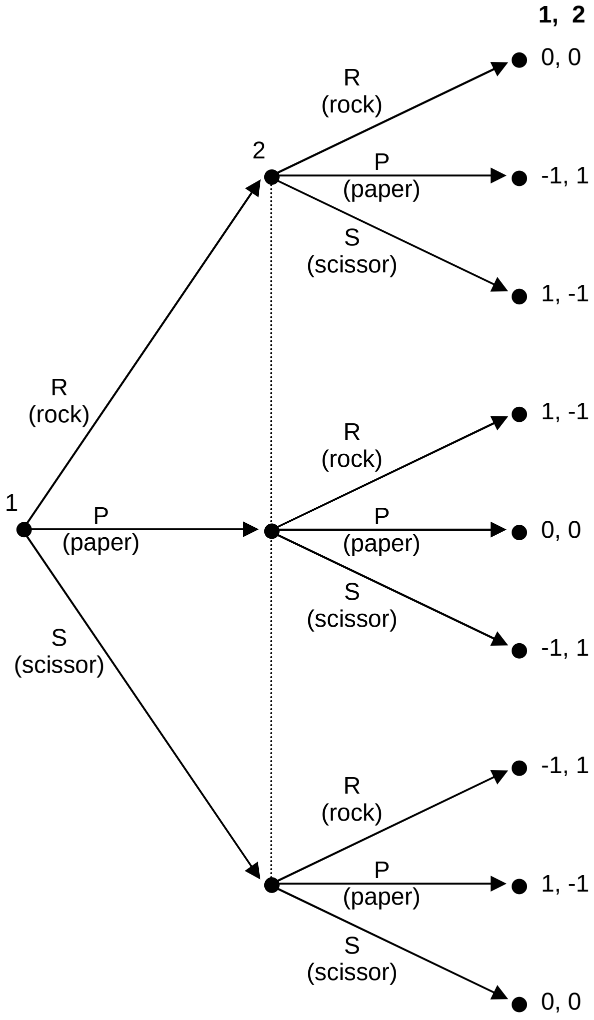
(b)



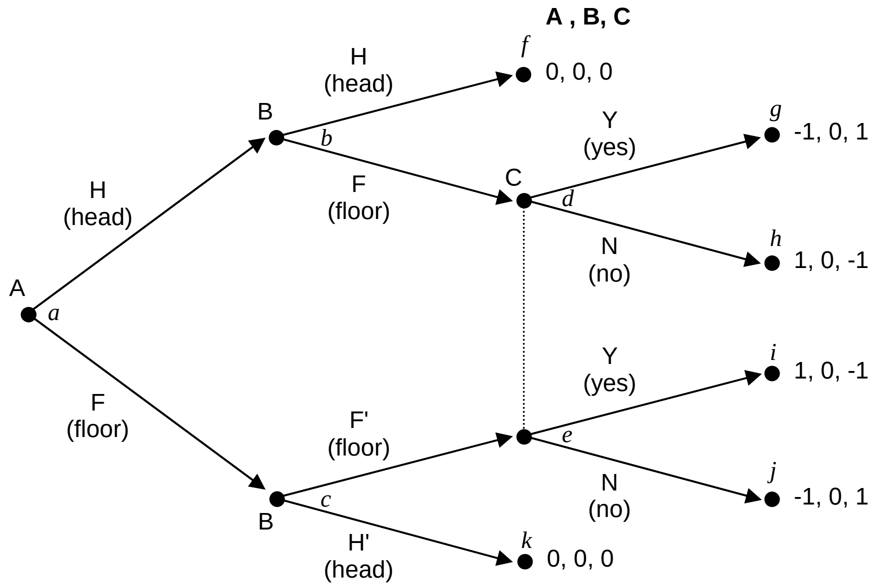
2.3. Consider the following strategic situation concerning the owner of a firm (O), the manager of the firm (M), and a potential worker (W). The owner first decides whether to hire the worker, to refuse to hire the worker, or to let the manger make the decision. If the owner lets the manager make the decision, then the manager must choose between hiring the worker or not hiring the worker. If the worker is hired then he or she chooses between working diligently and shirking. Assume that the worker does not know whether he or she was hired by the manager or the owner when he or she makes this decision. If the worker is not hired, then all three payers get a payoff of 0. If the worker is hired and shirks, then the owner and manager each get a payoff of -1, whereas the worker gets 1. If the worker is hired by the owner and works diligently, then the owner gets a payoff of 1, the manager gets 0, and the worker gets 0. If the worker is hired by the manager and works diligently, then the owner gets 0, the manager gets 1, and the worker gets 1. Represent this game in the extensive form (draw the game tree).



2.7. The following game is routinely played by youngsters--and adults as well--throughout the world. Two players simultaneously throw their right arms up and down to the count of "one, two, three." (Nothing strategic happens as they do this.) On the count of three, each player quickly forms his or her hand into the shape of either a rock, a piece of paper, or a pair of scissors. Abbreviate these shapes as R, P, and S, respectively. The players make this choice at the same time. If the players pick the same shape, then the game ends in a tie. Otherwise one of the players wins and the other loses. The winner is determined by the following rule: rock beats scissors, scissors beats paper, and paper beats rock. Each player obtains a payoff of 1 if he or she wins, -1 if he or she loses, and 0 if he or she ties. Represent this game in the extensive form.



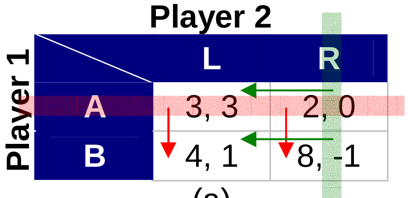
2.9. Consider the following strategic setting. There are three people: Amy, Bart, and Chris. Amy and Bart have hats. These three people are arranged in a room so that Bart can see everything that Amy does, Chris can see everything that Bart does, but the players can see nothing else. In particular, Chris cannot see what Amy does. First, Amy chooses either to put her hat on her head (abbreviated by H) or on the floor (F). After observing Amy's move, Bart chooses between putting his hat on his head or on the floor. If Bart puts his hat on his head, the game ends and everyone gets a payoff of 0. If Bart puts his hat on the floor, then Chris must guess whether Amy's hat is on her head by saying either "yes" or "no." This ends the game. If Chris guesses correctly, then he gets a payoff of 1 and Amy gets a payoff of -1. If he guesses incorrectly, then these payoffs are reversed. Bart's payoff is 0, regardless of what happens. Represent this game in the extensive form (draw the game tree).



Documentation.

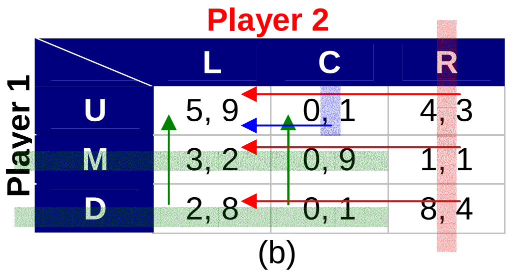
Prof Slutski went over simultaneous infinite decision (2.1b) in class. I checked my answers with Guille Sabbioni.

6.1. Determine which strategies are dominated in the following normal-form games.



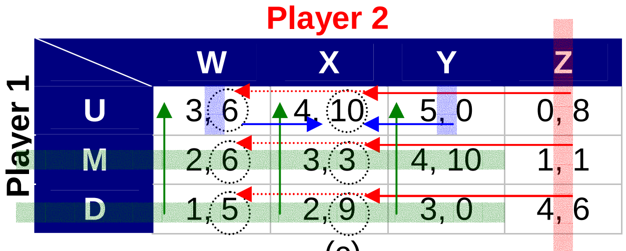
	L	R
A	3, 3	2, 0
B	4, 1	8, -1

(a)



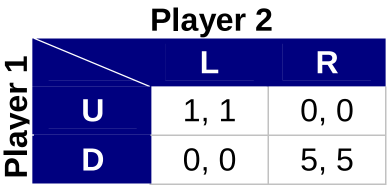
	L	C	R
U	5, 9	0, 1	4, 3
M	3, 2	0, 9	1, 1
D	2, 8	0, 1	8, 4

(b)



	W	X	Y	Z
U	3, 6	4, 10	5, 0	0, 8
M	2, 6	3, 3	4, 10	1, 1
D	1, 5	2, 9	3, 0	4, 6

(c)



	L	R
U	1, 1	0, 0
D	0, 0	5, 5

(d)

(a) B dominates **A**; L dominates **R**

(b) L dominates **R**; with R gone, U dominates **M** and **D**; with M and D gone, L dominates **C**

(c) 0.5W & 0.5X dominates **Z**; with Z gone, U dominates **M** and **D**; with M and D gone, X dominates **W** and **Y**

(d) **No** strategies are dominant (although (U,L) and (D,R) are Nash equilibria)

6.3. Consider a version of the Cournot duopoly game, where firms 1 and 2 simultaneously and independently select quantities to produce in a market. The quantity selected by firm i is denoted q_i and must be greater than or equal to zero, for $i = 1, 2$. The market price is given by $p = 100 - 2q_1 - 2q_2$. Suppose that each firm produces at a cost of 20 per unit. Further, assume that each firm's payoff is defined as its profit. Is it ever a best response for player 1 to choose $q_1 = 25$? Suppose that player 1 has the belief that player 2 is equally likely to select each of the quantities 6, 11, and 13. What is player 1's best response?

(a) Profit: $\pi^1 = pq_1 - 20q_1 = (100 - 2q_1 - 2q_2)q_1 - 20q_1 = 100q_1 - 2q_1^2 - 2q_1q_2 - 20q_1 = 80q_1 - 2q_1^2 - 2q_1q_2$

Best Reply for Firm 1: solve $\partial\pi^1/\partial q_1 = 0$ for q_1

$\partial\pi^1/\partial q_1 = 80 - 4q_1 - 2q_2 = 0$

$R^1(q_2) = 20 - {}^1/_2\, q_2$

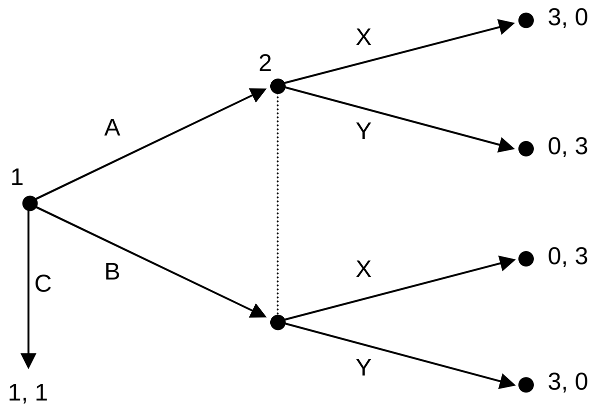
$R^1(q_2) = 20 - {}^1/_2\, q_2 = 25 \Rightarrow q_2 = -10$ which is infeasible $\therefore$ it is **never** a best response for player 1 to choose $q_1 = 25$

(b) $E(q_2) = (6 + 11 + 13)/3 = 10$

$R^1(E(q_2)) = 20 - {}^1/_2\, 10 =$ **15**

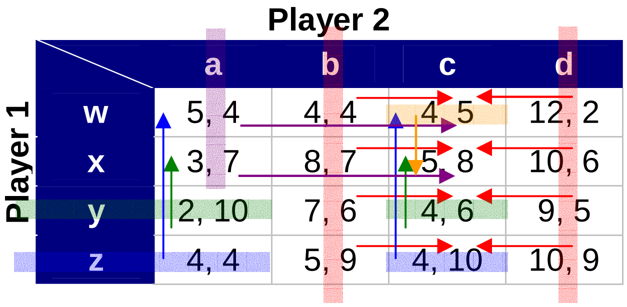
This is the same as $E(R^1(q_2)) = [(20 - {}^1/_2(6)) + (20 - {}^1/_2(11)) + (20 - {}^1/_2(13))]/3 = 15$

7.3. In the game pictured here, is it ever rational for player 1 to select strategy C? Why?



No. Using a mixed strategy of A with probability 0.5 and B with probability 0.5, player 1's expected payoff is 1.5 so it dominates strategy C.

7.5. Find the set of rationalizable strategies for the following game.



		Player 2			
		a	**b**	**c**	**d**
Player 1	**w**	5, 4	4, 4	4, 5	12, 2
	x	3, 7	8, 7	5, 8	10, 6
	y	2, 10	7, 6	4, 6	9, 5
	z	4, 4	5, 9	4, 10	10, 9

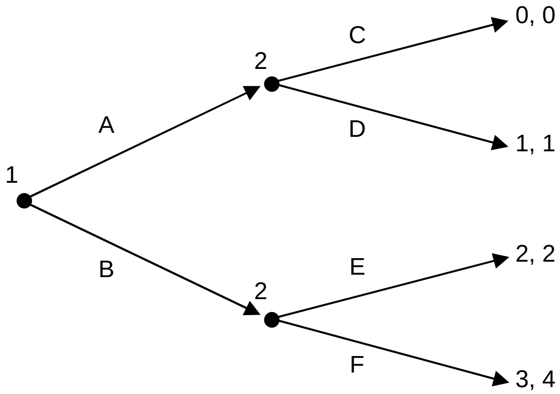
Note that each player has more than one dominated strategy. Discuss why, in the iterative process of deleting dominated strategies, the order in which dominated strategies are deleted does not matter.

c dominates **b** and **d**; with b and d gone x dominates **y** and w dominates **z**; with y and z gone, c dominates **a**; with a gone x dominates **w**; ∴ the set of rationalizable strategies only has strategy x for player 1 and strategy c for player 2: **{(c,x)}**

By definition a dominated strategy is one that a rational player would not use. Strategies are dominated by non-dominated strategies. Saying that the order of deleting strategies matters suggests that a dominated strategy (say B dominated by A) is used to remove another dominated strategy (say C). But that first dominated strategy (B) is dominated by A which also dominates C. Therefore, we could just as easily delete C based on domination by A or domination by B. Strategy B will then be deleted because it's dominated by A. The order doesn't matter.

9.1. Find the Nash equilibria of the games in the figures in Exercise 3(a-d) at the end of Chapter 4. Remember to convert these games into the normal form first.

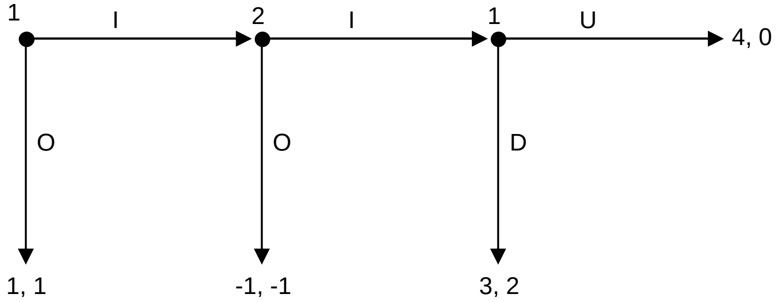
(a) Nash equilibria: **{(B, CF), (B, DF)}**; Note: that means player 1 plays strategy B and player 2 plays strategy F.



Player 2

		CE	CF	DE	DF
Player 1	**A**	0, 0	0, 0	1, 1	1, 1
	B	2, 2	3, 4	2, 2	3, 4

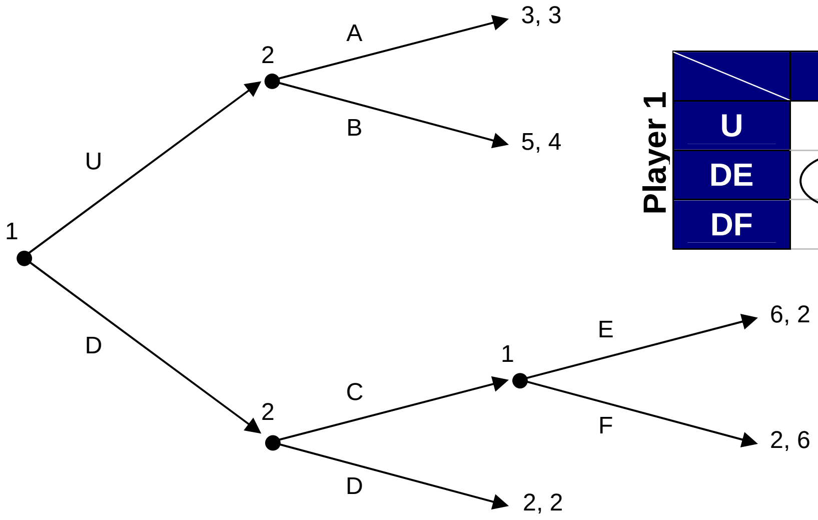
(b) Nash equilibra: **{(IU, I), (O, O)}**; Note: (IU, I) is strict, but (O, O) isn't. We could look at OU and OI for player 1, but these is redundant and not necessary.



Player 2

		I	O
Player 1	**IU**	4, 0	-1, -1
	ID	3, 2	-1, -1
	O	1, 1	1, 1

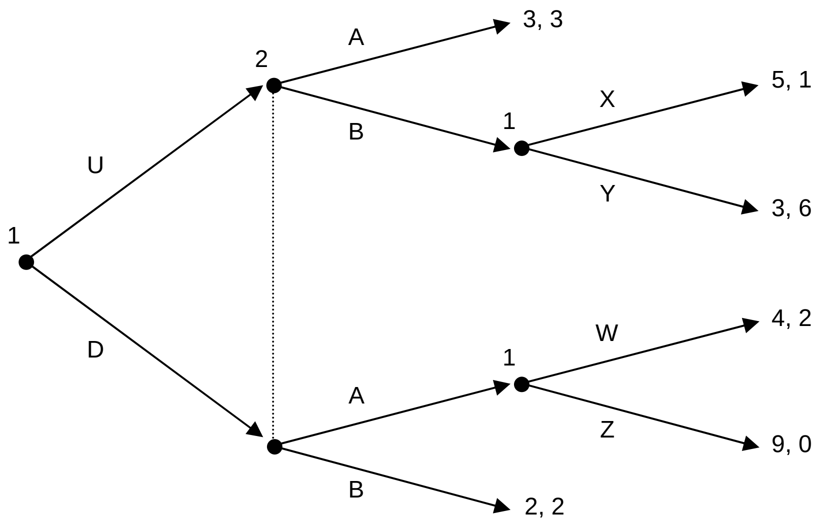
(c) Nash equilibria: **{(DE, AC), (DE, BC), (U, BD)}** ; Note: that means player 1 plays strategy D and player 2 plays strategy C, or player 1 plays strategy U and player two plays strategy B. We could look at UE and UF for player 1, but these are redundant and not necessary.



Player 2

		AC	AD	BC	BD
Player 1	**U**	3, 3	3, 3	5, 4	5, 4
	DE	6, 2	2, 2	6, 2	2, 2
	DF	2, 6	2, 2	2, 6	2, 2

(c) Nash equilibria: **none**. We could add strategies for player 1 (UXW, UXZ, UYW, UXZ, DXW, DXZ, DYW, DYZ), but that doesn't fundamentally change the problem; it just copies the four rows here because X & Y are irrelevant when player 1 plays D and W & Z are irrelevant when player 1 plays U.



		Player 2	
		A	**B**
Player 1	**UX**	3, $\underline{3}$	$\underline{5}$, 1
	UY	3, $\underline{3}$	3, $\underline{6}$
	DW	4, 2	2, 2
	DZ	$\underline{9}$, 0	2, 2

9.5. Find the Nash equilibrium of the following normal-form game: $S_1 = [0, 1]$, $S_2 = [0, 1]$, $u_1(s_1, s_2) = 3s_1 - 2s_1s_2 - 2s_1^2$, and $u_2(s_1, s_2) = s_2 + 2s_1s_2 - 2s_2^2$. (The solution is interior so you can use calculus.)

Best Reply Functions:

$$\frac{\partial u_1(s_1,s_2)}{\partial s_1} = 3 - 2s_2 - 4s_1 = 0 \Rightarrow s_1 = {}^3/_4 - {}^1/_2 \cdot s_2$$

$$\frac{\partial u_2(s_1,s_2)}{\partial s_2} = 1 + 2s_1 - 4s_2 = 0 \Rightarrow s_2 = {}^1/_4 + {}^1/_2 \cdot s_1$$

Nash equilibrium is when each player is playing best reply to opponent's strategy (i.e., intersection of best reply functions):

$s_1 = {}^3/_4 - {}^1/_2({}^1/_4 + {}^1/_2 \cdot s_1) = {}^3/_4 - {}^1/_8 - {}^1/_4 \cdot s_1 \Rightarrow s_1 = {}^4/_5({}^5/_8) = {}^4/_8 = \mathbf{{}^1/_2}$

$s_1 = {}^1/_4 + {}^1/_2({}^1/_2) = \mathbf{{}^1/_2}$

10.1. Consider a more general Cournot model than the one presented in this chapter. Suppose there are n firms. The firms simultaneously and independently select quantities to bring to the market. Firm i's quantity is denoted q_i, which is constrained to be greater than or equal to zero. All of the units of the good are sold, but the prevailing market price depends on the total quantity in the industry, which is $Q = \sum_{i=1}^{n} q_i$. Suppose the price is given by $p = a - bQ$ and suppose each firm produces with marginal cost c. There is no fixed cost for the firms. Assume $a > c > 0$ and $b > 0$. Note that firm i's profit is given by $u_i = p(Q)q_i - cq_i = (a - bQ)q_i - cq_i$. Defining Q_{-i} as the sum of the quantities produced by all firms except firm i, we have $u_i = (a - bq_i - bQ_{-i})q_i - cq_i$. Each firm maximizes its own profit.

(a) Represent this game in the normal form by describing the strategy spaces and payoff functions.

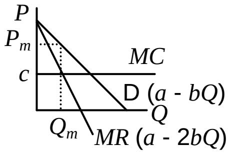
(b) Find firm i's best response function as a function of Q_{-i}. Graph this function.

(c) Compute the Nash equilibrium of this game. Report the equilibrium quantities, price, and total output. Hint: Summing the best-response functions over the different players will help.

(d) Show that, for the Cournot duopoly game ($n = 2$), the set of rationalizable strategies coincides with the Nash equilibrium.

(a) Players: $I = \{1, 2, ..., n\}$

Strategy Spaces: **S^i = [0, ∞)** ($\forall\ i \in I$) because player i can't produce a negative quantity. The upper bound could be tightened to a more realistic quantity in two ways. First, realize that it doesn't make sense for anyone to produce to the point that prices go negative. Therefore, solving $a - bQ = 0$ yields an upper bound of **a/b**. A tighter bound comes from realizing that a firm producing by itself will behave like a monopoly. That means $MR = a - 2bQ$. Solving $MR = MC = c$ yields an upper bound of **$(a - c)/2b$**.

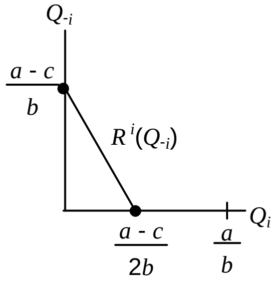


Payoff Functions: **$u_i(q_1, q_2, ..., q_n) = (a - bq_i - bQ_{-i})q_i - cq_i$**, where $q_i \in S^i\ \forall\ i \in I$

(b) Best Response for firm i: $\dfrac{\partial u_i}{\partial q_i} = a - 2bq_i - bQ_{-i} - c = 0 \ \Rightarrow$

$$\boxed{R^i(Q_{-i}) = \frac{a-c}{2b} - \frac{Q_{-i}}{2}}$$

Intercepts: $Q_{-i} = 0 \ \Rightarrow \ q_i = \dfrac{a-c}{2b}$; $q_i = 0 \ \Rightarrow \ \dfrac{a-c}{b}$



(c) Nash equilibrium means players play best responses to opponents' best responses.

$$q_i^* = \frac{a-c}{2b} - \frac{Q_{-i}^*}{2}$$

Sum all q_i^* to get total output, Q^*

$$Q^* = \sum_{i=1}^{n} q_i^* = \sum_{i=i}^{n}\left(\frac{a-c}{2b} - \frac{Q_{-i}^*}{2}\right) = \frac{n(a-c)}{2b} - \frac{1}{2}\sum_{i=i}^{n} Q_{-i}^*$$

Tricky part... sub $Q_{-i} = \sum_{j \neq i} q_j$ and realize $\sum_{i=i}^{n}\sum_{j \neq i} q_j = (n-1)\sum_{i=1}^{n} q_i$

$$\sum_{i=1}^{n} q_i^* = \frac{n(a-c)}{2b} - \frac{n-1}{2}\sum_{i=i}^{n} q_i^*$$

Move sums to same side

$$\left(1 + \frac{n-1}{2}\right)\sum_{i=1}^{n} q_i^* = \frac{n(a-c)}{2b}$$

Solve for $Q*$

$$Q* = \sum_{i=1}^{n} q_i * = \frac{n(a-c)}{2b}\left(\frac{2}{n+1}\right) = \boxed{\frac{n(a-c)}{b(n+1)}}$$

Substitute $Q*$ to solve for $p*$

$$p* = a - bQ* = a - b\left(\frac{n(a-c)}{b(n+1)}\right) = \frac{an+a}{n+1} - \frac{na-nc}{n+1} = \boxed{\frac{a+nc}{n+1}}$$

Substitute $Q*$ to solve for $q_i *$

$$q_i * = \frac{a-c}{2b} - \frac{Q_{-i} *}{2}$$

Trick is to substitute $Q_{-i} * = Q* - q_i *$

$$q_i * = \frac{a-c}{2b} - \frac{Q* - q_i *}{2} = \frac{a-c}{2b} - \frac{1}{2}\left(\frac{n(a-c)}{b(n+1)}\right) + \frac{q_i *}{2}$$

Put all $q_i *$ on same side

$$\frac{1}{2} q_i * = \frac{na+a-nc-c}{2b(n+1)} - \frac{na-nc}{2b(n+1)} = \frac{a-c}{2b(n+1)}$$

Solve for $q_i *$

$$q_i * = \left(\frac{a-c}{2b(n+1)}\right)2 = \boxed{\frac{a-c}{b(n+1)}}$$

(d) $n = 2$

$$Q* = \frac{2(a-c)}{3b};\ p* = \frac{a+2c}{3};\ q_i * = \frac{a-c}{3b} = \frac{Q*}{2}$$

From textbook problem (p.96) $a = 1000$, $b = 1$, and $c = 100$

$$Q* = \frac{2(1000-100)}{3(1)} = 600;\ p* = \frac{1000+2(100)}{3} = 400;\ q_i * = \frac{1000-100}{3(1)} = 300$$

Which is the same solution given on p.96.

Documentation.

I reviewed my work with Josh Kneifel. He caught an error in 6.1.a (my drawing figure didn't match my verbiage). We also worked out problem 10.1 together. Ozde Oztekin also caught an error in 6.1b and d (I didn't copy the problem correctly).