Introduction to Economic Growth

Development of Modern Growth Theory

Questions -

1. Why are some people (countries) rich and some poor?
   Need to find determinants of GDP and growth rates to answer this
2. Why do some countries grow faster than others?
3. In long-run, is growth rate exogenous or endogenous?

Endogenous Variable - Determined by decisions (usually based on optimizing behavior; e.g., maximize utility leads to demand; maximize profit leads to supply); (2) something you can change through policy (e.g., tax will change P_D, P_S, and Q)

Supply & Demand - $P_D = a - bQ$ and $P_S = c + dQ$; this leads to $P_D = P_S = a - bQ = c + dQ$ $\Rightarrow$ $Q = (a - c)/(d + b)$ $\therefore$ Q is <u>given</u> (endogenous)

Growth Rate - can policies affect growth rate (i.e., is growth rate endogenous)? neoclassical theory says yes, but only in transition, not in long-run; some newer theories say yes; others say no

Technology - technical description: production function; intuitive: instructions for how to produce things

Rice Example - cooking rice requires a pot and heat source (capital), rice and water (resources), and a recipe with directions on how to cook it (technology)

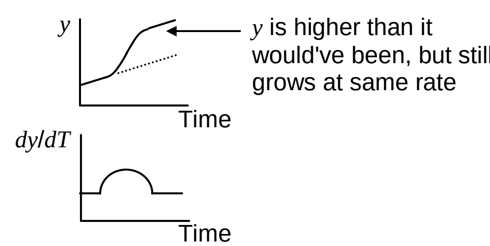
Steady State Equilibrium - also called long-run equilibrium; all variables grow at a constant rates (could be different from each other, but constant over [or independent of] time)

Brief History -

Adam Smith - focused on accumulation of capital
Carl Marx - focused on technological change and unemployment
Ricardo - focused on class structure
Neoclassical Growth Theory - <u>Solow</u> (1956); assumptions:

- Perfect competition in all markets
- Exogenous rate of population growth (in long-run)
- Exogenous rate of technological change (in long-run)
- Endogenous income per capita levels
- Endogenous transitional growth rates

Example - look at income per capita (y); we can change the growth rate (dy/dT) in transition, but not in the long-run (i.e., slope stays the same)



New Growth Theory - <u>Romer</u> (1986) & <u>Lucas</u> (1988); also called **endogenous growth** theory because now long-run growth rate can change; dropped perfect competition assumption and made technical change endogenous

Schumpeterian Approach to Economic Growth - new products have short lifetimes (all companies from the original Dow Industrial list are gone except General Electric); firms enjoy temporary monopoly power (imperfect competition) then are replaced (creative destruction); all used fixed population (not realistic) because introducing any population growth means economic growth unlimited (also not realistic)

<u>Grossman & Hylpman</u> (1991), <u>Aghion & Howitt</u> (1992), <u>Segerstrom</u>, et. al. (1990) - focused on product quality

<u>Romer</u> (1990) - focused on product variety

Schumpeterian Without Scale Effects - original Schumpeterian approach had scale effects which made growth rate change with population rate (e.g., double population caused growth to double); modification without scale effects fixed the problem
Jones (1995); Segerstrom (1990) - exogenous long-run growth
Howitt (1999); Dinopoulos & Thompson (1999) - endogenous long-run growth

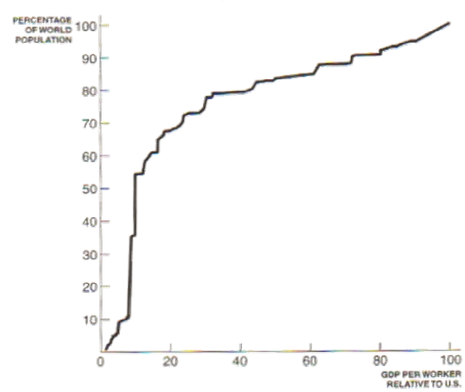
Facts on Economic Growth

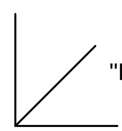
1. There is **enormous variation in per capita income across economies**. The poorest counties have per capita incomes that are less then 5 percent of per capita incomes in the richest countries
    - **GDP** - gross domestic product; monetary value of all goods and services produced within a nation's borders in 1 year; doesn't matter which company does it (e.g., Honda Accords built in Ohio count towards US GDP)
    - **GNP** - gross national product; similar to GDP except it's goods and services produces by a nation's companies (e.g., Honda Accords built in Ohio count towards Japan's GNP)
        - **Difference** - GDP better to use because it reflects quality of life more than GNP, but in long-run both are very similar
    - **Purchasing Power Parity** - used to develop exchange rates for GDP comparisons; attempts to measure actual value of a currency in terms of its ability to purchase similar products (e.g., Big Mac in U.S. costs \$2 and in Japan costs 300 yen ⇒ PPP exchange rate is 150 yen/\$)
    - **Per Capita Income** - GDP per person; useful "summary statistic" of the level of economic development because it's highly correlated with other measures of quality of life
    - **GDP Per Worker** - tells more about productivity of the labor force (whereas GDP per person is general measure of welfare)
    - Why is there a difference?

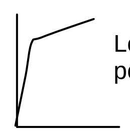
TABLE 1.1 STATISTICS ON GROWTH AND DEVELOPMENT

	GDP per capita, 1997	GDP per worker, 1997	Labor force participation rate, 1997	Average annual growth rate, 1960–97	Years to double
"Rich" countries					
U.S.A.	$20,049	$40,834	0.49	1.4	50
Japan	16,003	25,264	0.63	4.4	16
France	14,650	31,986	0.46	2.3	30
U.K.	14,472	29,295	0.49	1.9	37
Spain	10,685	29,396	0.36	3.5	20
"Poor" countries					
China	2,387	3,946	0.60	3.5	20
India	1,624	4,156	0.39	2.3	30
Zimbabwe	1,242	2,561	0.49	0.4	192
Uganda	697	1,437	0.49	0.5	146
"Growth miracles"					
Hong Kong	18,811	28,918	0.65	5.2	13
Singapore	17,559	36,541	0.48	5.4	13
Taiwan	11,729	26,779	0.44	5.6	12
South Korea	10,131	24,325	0.42	5.9	12
"Growth disasters"					
Venezuela	6,760	19,455	0.35	−0.1	−517
Madagascar	577	1,334	0.43	−1.5	−46
Mali	535	1,115	0.48	−0.8	−85
Chad	392	1,128	0.35	−1.4	−48

FIGURE 1.1 CUMULATIVE DISTRIBUTION OF WORLD POPULATION BY GDP PER WORKER, 1995




"Fair"


Lots of poor

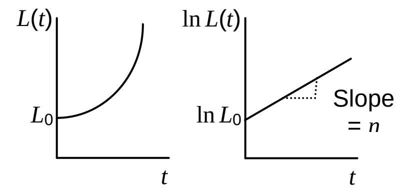
2. **Rates of economic growth vary substantially across countries**.

   Growth - the amount by which something changes (first derivative)

   Growth Rate - amount by which something changes divided by initial value (first derivative over variable)

   Constant Growth Rate - measure of interest grows exponentially at a constant rate; graph looks exponential, but on a $\ln$ scale it's linear with slope equal to the growth rate

   Example - $L(t)$ = labor force at time t; labor growth = $L'(t)$; labor growth rate = n; we can relate these by definition: $L'(t)/L(t) = n$

   Now solve for $L(t)$ by writing out $L'(t)$ as dL/dt: $dL/dt/L(t) = n$

   Move dt to right side and integrate: $\int dL/L(t) = \int n dt$

   Work out integrals: $\ln L(t) = nt + c$ (constant)

   Use exponential: $e^{\ln L(t)} = L(t) = e^{nt+c} = e^c e^{nt}$

   Notice that using $t = 0$, means that $L(0) = e^c$

   Rewrite $L(0)$ as L_0 and we solved for $L(t)$ in terms of L_0 and growth rate n:

   $$L(t) = L_0 e^{nt}$$

   (This is general equation for any **constant growth rate**)

   Going the other way - differentiate to find $L'(t) = L_0 e^{nt} \cdot d(nt)/dt = nL_0 e^{nt}$

   Use growth rate definition: $L'(t)/L(t) = nL_0 e^{nt}/L_0 e^{nt} = n$

   Yet another way - take $\ln$ of both sides: $\ln L(t) = \ln L_0 + nt$ (used for regression)

   Now take derivative of both sides wrt t: $\partial \ln L(t)/\partial t = \partial \ln L_0/\partial t + \partial nt/\partial t$

   $L'(t)/L(t) = 0 + n = n$

   Years to Double - useful way to interpret growth rates introduced by Robert Lucas ("On the Mechanics of Economic Development," 1988); solve for t in order to double income $Y(t)$ from Y_0: $2Y_0 = Y_0 e^{gt} \Rightarrow 2 = e^{gt}$

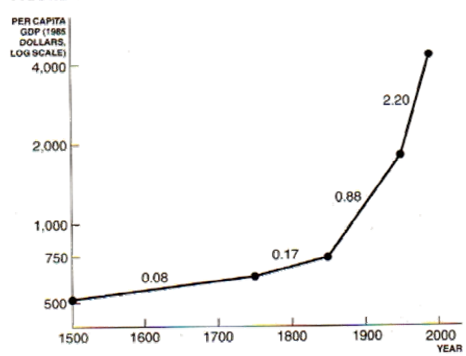
   Take $\ln$ of both sides: $\ln 2 = gt$

   Solve for t: $$t = \ln 2/g \cong 0.7/g$$

   $\therefore$ country growing at g percent per year will double per capita income every $70/g$

3. **Growth rates are not generally constant over time**. For the world as a whole, growth rates were close to zero over most of history but have increased sharply in the twentieth century. For individual countries, growth rates also change over time.

   FIGURE 1.3 WORLD PER CAPITA GDP AND GROWTH RATES, 1500-1990

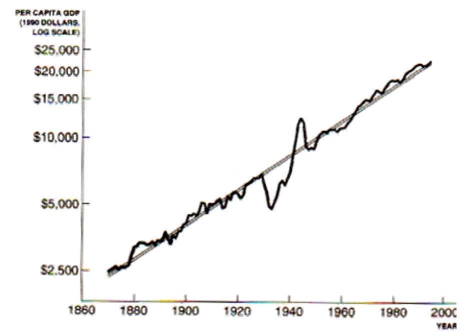
   Note: some growth models don't allow growth rate to change in long-run, just in transition

   Dismal Science - prior to higher rates of growth, Thomas Malthus argued that population growth was exponential, but food production was linear; he argued that we would have endless wars to control population growth (coined term "dismal science" for economics)

4. A country's relative position in the world distribution of per capital incomes is **not immutable**. Countries can move from being "poor" to being "rich," and vice-versa.

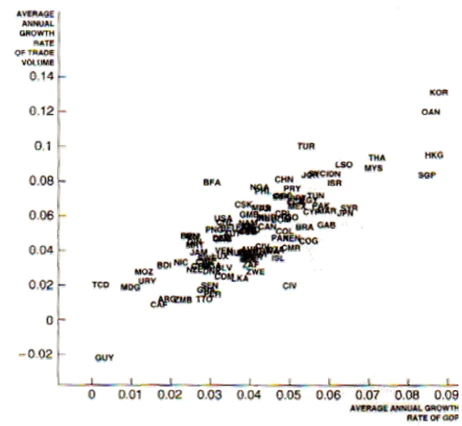
5. In the United States (general for most economics "in the long run") over the last century,
   a. rate of return to capital, r, shows no trend upward or downward
      r constant ($\cong$ 3%) based on interest rate on government debt
   b. shares of income devoted to capital, rK/Y, and labor, wL/Y, show no trend
      Labor constant ($\cong$ 70% GDP $\Rightarrow$ capital $\cong$ 30% GDP); also, labor 50-50 between skilled and unskilled
   c. average growth rate of output per person has been positive and relatively constant over time; i.e., the United States exhibits steady, sustained per capita income growth
      g constant ($\cong$ 1.8%)

FIGURE 1.4 REAL PER CAPITA GDP IN THE UNITED STATES, 1870–199



6. Growth in output and growth in the volume of international trade are closely related. Strong positive correlation between international trade and growth (i.e., nations that trade more have higher GDP)
   Volume of Trade - sum of exports and imports

FIGURE 1.5 GROWTH IN TRADE AND GDP, 1960–90



Neoclassical Growth Models

Model - mathematical representation of some aspect of the economy; best models are often very simple but convey enormous insight into how the world works

"All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive." - Robert Solow, 1956

Solow Model

Assumptions

1. Single, **homogeneous good** (output) - this also implies that there is no international trade (have to have at least two different goods for any trade to take place)
2. **Technology is exogenous** - technology available to firms is unaffected by the actions of the firms, including R&D (we'll relax this later); implication is that production function is not shifting
3. **Keynesian Assumption** - individuals save a constant fraction of their income (i.e., people consume in proportion to income)... this assumption is consistent with the data... let the savings rate be s $\therefore$ total savings is sY
4. **Labor Force** - population is same as labor force (good enough to have constant labor force participation rate); population grows at constant rate n $\therefore$ $L(t) = L_0 e^{nt}$
5. **Perfect Competition** - zero economic profit; price takers in labor market (w) and capital market (r)
6. **Constant Returns to Scale** - can use any constant returns production function (i.e., double input produces double output); we'll focus on Cobb-Douglas
   Two Inputs - capital and labor; capital is accumulated endogenously through savings
   Constant Elasticity of Substitution (CS) - production function with constant returns to scale; generalization of Cobb-Douglas; $Y(t) = F(K,L) = [bK^{\rho} + cL^{\rho}]^{1/\rho}$
   Cobb-Douglas Function - $Y(t) = F(K,L) = K^{\alpha}L^{1-\alpha}$, $\alpha \in (0,1)$
      Test Constant Returns - $F(2K,2L) = (2K)^{\alpha}(2L)^{1-\alpha} = 2K^{\alpha}L^{1-\alpha} = 2F(K,L)$... yep
7. **Constant Depreciation Rate** - capital depreciates at a constant rate δ

Basic Model - built around two equations: production function & capital accumulation equation

I. Production Function - describes how capital inputs, $K(t)$, combine with labor, $L(t)$, to produce output, $Y(t)$

Capital - look at corn... some you eat (consumption), some you plant to eat more in the future (capital); capital postpones consumption by certain amount in hopes of higher (but uncertain) future consumption... definition of investment

Max Profits - hire capital and labor at market prices to maximize profits (**Note:** we can consider our good to be a numeraire (P = 1) which means r and w are in terms of Y)
Max $F(K,L) - rK - wL$

First Order Conditions -

$\frac{d\pi}{dK} = F_K - r = 0$ ... marginal product of capital = r

$\frac{d\pi}{dL} = F_L - w = 0$ ... marginal product of labor = w

Using Cobb-Douglas -

$F_K = \alpha K^{\alpha-1} L^{1-\alpha} = r \;\; \Rightarrow \;\; r = \frac{\alpha K^{\alpha} L^{1-\alpha}}{K} = \frac{\alpha Y}{K}$

$F_L = (1-\alpha) K^{\alpha} L^{(1-\alpha)-1} = w \;\; \Rightarrow \;\; w = \frac{(1-\alpha) K^{\alpha} L^{1-\alpha}}{L} = \frac{(1-\alpha)Y}{L}$ ... we can use this to get the demand for labor

Finding Steady State - in order to find some form of the production function to give us a steady state, we need to identify variables that won't change over time; in this case, we can focus on r & w because (1) empirical data says they're stable over time [fact 5 from Introduction to Growth], (2) based on the way the production function is set up, if either of these variables grows, the other has to decline and if we're talking growth rates, that means the other variable will eventually reach zero... not realistic

From first order conditions we have $w = \frac{(1-\alpha)Y}{L} \;\; \therefore \;\; \frac{Y}{L}$ is also constant; that's called the **output per worker**: $y \equiv \frac{Y}{L}$

We also have $r = \frac{\alpha Y}{K} \;\; \therefore \;\; \frac{Y}{K}$ is also constant; to put things in same terms (per worker or per capita), we'll use a trick: $\frac{Y/L}{K/L} = \frac{y}{k}$; since y is constant that means **capital per worker**: $k \equiv \frac{K}{L}$ is also constant

Results for Production Function -

1. **Zero Profit** - results from constant returns assumption;

   Profit = $Y - rK - wL = Y - \frac{\alpha Y}{K} K - \frac{(1-\alpha)Y}{L} = Y - \alpha Y - (1-\alpha)Y = 0$

   $\therefore$ payments to inputs completely exhaust the value of output produced
2. **Constant Labor & Capital** - as fractions of GDP; results from Cobb-Douglas production function: $\frac{rK}{Y} = \alpha$ and $\frac{wL}{Y} = 1-\alpha$; this agrees with empirical data (see fact 5 from Introduction to Growth)
3. **Growth Rates Equal** - take $\ln$ of both sides of capital per worker: $k = K/L$

   $\ln k = \ln K - \ln L$

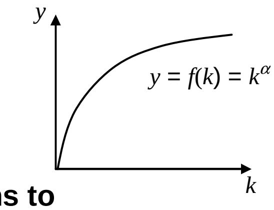
   Totally differentiate wrt t: $\frac{1}{k}\frac{dk}{dt} = \frac{1}{K}\frac{dK}{dt} - \frac{1}{L}\frac{dL}{dt}$ or $\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L}$

   Those are growth rates; k is a stable variable in steady state so dk/dt = 0; that means $\frac{\dot{K}}{K} = \frac{\dot{L}}{L}$ ... the growth rate of capital equals the growth rate of labor in the steady state

Per Capita Production - now that we have steady state variables we can work with (k and y), we need to convert the production function to incorporate these variables:

$Y = F(K, L)$ ... because of constant returns to scale, we can write

$y = \frac{Y}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right) = F(k,1)$ ... introduce per capita production function f: $y = f(k)$



Using Cobb-Douglas - $y = f(k) = \frac{K^{\alpha} L^{1-\alpha}}{L} = K^{\alpha} L^{-\alpha} = \left(\frac{K}{L}\right)^{\alpha} = k^{\alpha}$

$y' = \alpha k^{\alpha-1} > 0$ $\therefore$ output per worker always increases with k increases

$y'' = \alpha(\alpha-1)k^{\alpha-2} < 0$ (because $\alpha < 1$) $\therefore$ we have **diminishing returns to capital** (the added output per worker by increasing capital per worker declines as we add more capital)

II. Capital Accumulation - the change in the capital stock over time is equal to the infusion of new capital from savings (sY) minus the depreciation of capital

$$\frac{dK}{dt} = \dot{K} = sY - \delta K$$

Example - economy starts with output of 100 and capital base of 200; the capital depreciate rate is 5 percent and the savings rate is 30 percent

Savings (gross investment) = sY = 0.3(100) = 30

Depreciation = δK = 0.05(200) = 10

Capital accumulation (net investment) = sY - δK = 30 - 10 = 20

Per Capita Capital Accumulation - just like we did with the production function, we want to get this equation to incorporate variables that are constant in the steady state (k and y)

Divide all terms by K: $\frac{dK/dt}{K} = s\frac{Y}{K} - \delta\frac{K}{K} = s\frac{Y}{K} - \delta$

Where did we see that $\frac{dK/dt}{K}$ term before? We got it from differentiation

$\ln k = \ln K - \ln L$ wrt t when we showed that the growth rates of capital and labor are equal in the steady state (on previous page); the general (non steady state) result was $\frac{dk/dt}{k} = \frac{dK/dt}{K} - \frac{dL/dt}{L}$

From assumption 4 substitute the labor growth rate n: $\frac{dk/dt}{k} = \frac{dK/dt}{K} - n$

Solve that for $\frac{dK/dt}{K}$: $\frac{dK/dt}{K} = \frac{dk/dt}{k} + n$

Substitute that into the first equation we had: $\frac{dk/dt}{k} + n = s\frac{Y}{K} - \delta$

To get y into the equation, use the L/L trick again: $\frac{dk/dt}{k} + n = s\frac{Y/L}{K/L} - \delta = s\frac{y}{k} - \delta$

Solve for capital accumulation per worker: $\frac{dk}{dt} = sy - (n+\delta)k$

Steady State - in steady state, k must be constant so dk/dt = 0 $\Rightarrow$ $\boxed{sy = (n+\delta)k}$

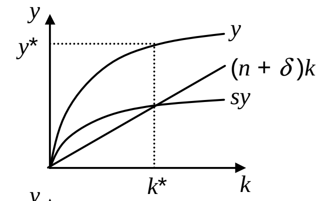
That is, the gross savings (investment in capital) per worker must equal the lost capital per worker (lost through depreciation and the increase in the number of workers)

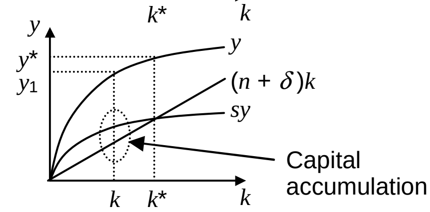
Solve for Growth Rates - use formulas with standard trick of taking $\ln$ of both sides, then differentiating

$$y = k^{\alpha} \;\Rightarrow\; \ln y = \alpha \ln k \;\Rightarrow\; \frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$$ ... since $\dot{k} = 0$, we must have $\dot{y} = 0$

$$Y = K^{\alpha} L^{1-\alpha} \;\Rightarrow\; \ln Y = \alpha \ln K + (1-\alpha)\ln L \;\Rightarrow\; \frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} =$$

$\alpha n + (1-\alpha) n = n$ ... so output grows at same rate as labor force





Graphically - the Solow model is pretty easy to solve graphically... just look at where sy intersects $(\delta + n)k$

Stable Equilibrium - if we have k anywhere other than k^*, it'll eventually adjust to be at k^*; for example, start with $k < k^*$; this means savings outpaces effective depreciation so we're accumulating capital ($k\uparrow$); this continues until $k = k^*$

Algebraically - solving is a little more complicated; start with stead state equation:

$$sy = (n+\delta)k$$

Plug in production per worker function:

$$sk^{\alpha} = (n+\delta)k$$

Solve for k:

$$k^{1-\alpha} = \frac{s}{n+\delta} \;\Rightarrow\; k = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

Plug that back into the production per worker function to solve for y:

$$y^* = (k^*)^{\alpha} = \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Advantage of Algebra - can check model empirically: $\ln y^* = \frac{\alpha}{1-\alpha}\ln s - \frac{\alpha}{1-\alpha}\ln(n+\delta)$ $\therefore$ regression should show parameters $\beta_1 = \beta_2$; also can solve for α and check if $\alpha < 1$

Summary - started with two equations:

$$Y = F(K, L) \quad \text{and} \quad \frac{dK}{dt} = sY - (n+\delta)K$$

Then introduced output per worker and capital per worker:

$$y \equiv Y/L \quad \text{and} \quad k \equiv K/L$$

Modified original equations:

$$y = f(k) \quad \text{and} \quad \frac{dk}{dt} = sy - (n+\delta)k$$

Steady state has $dk/dt = 0$ so we know

$$sy = (n+\delta)k$$

Other things we showed

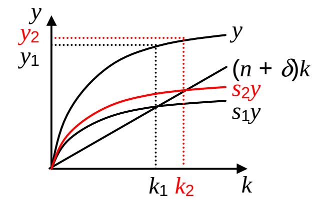
Growth rates of labor, capital, and output are equal: $\dot{L} = \dot{K} = \dot{Y} = n$

Measure	Steady State Growth Rate
y, k	0
L, Y, K	n

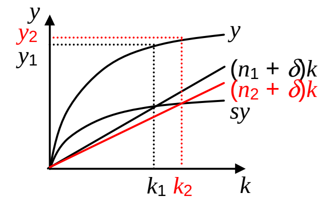
Comparative Statics

Increase Savings Rate - $s\uparrow \Rightarrow y\uparrow$ and $k\uparrow$

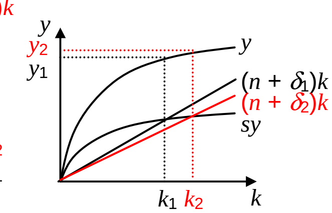
Note: growth rate in long run doesn't change since growth rate of capital must equal the growth rate of labor (population); the result will be a higher y (GDP per capita) that grows at the same rate as before the increased savings rate



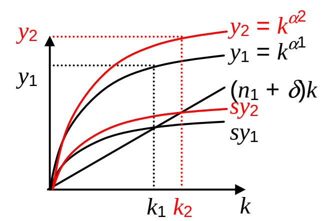
Decrease Population Growth Rate - $n\downarrow \Rightarrow y\uparrow$ and $k\uparrow$



Decrease Depreciation Rate - $\delta\downarrow \Rightarrow y\uparrow$ and $k\uparrow$



Increase Productivity of Capital - $\alpha\uparrow \Rightarrow y\uparrow$ and $k\uparrow$



Transitional Dynamics - how model evolves (between equilibria)

Algebraically - we already showed:

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$$

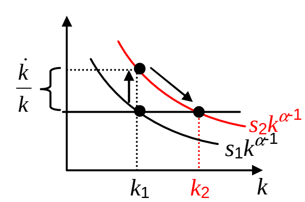
(which we used to find $\dot{y} = 0$ in steady state; we're not in steady state now so we have to work with the general version; the other equation we need is the capital per worker accumulation equation which we'll divide by k:

$$\frac{\dot{k}}{k} = \frac{sk^{\alpha}}{k} - (n+\delta) = \frac{s}{k^{1-\alpha}} - (n+\delta)$$

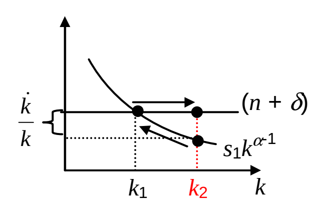
Now we have two differential equations to describe the transitional dynamics; since diffeq isn't fun, we'll just look at the transitions geometrically by plotting that second equation to determine what happens to the growth rate of k

Geometrically -

Increase Savings Rate - $s\uparrow$ shifts the $sk^{\alpha\text{-}1}$ curve; at the current capital per worker, we aren't in equilibrium so $\dot{k}/k$ increases (i.e., the rate of growth of capital per worker increases); note form the first equation we showed above that this means the output per worker is also increasing; eventually, we'll settle back down where $s_2k^{\alpha\text{-}1}$ = $n + \delta$ so $\dot{k}/k$ goes back to zero (so does $\dot{y}/y$); the end result is a one time increase in capital per worker (k) and output per worker (y), but the growth rate of these remains the same at zero



Decrease Population - if $L\downarrow$ (and n remains constant), we have a one time increase in capital per worker (k); this shifts us to the right on the graph and there's a disparity between ($n + \delta$) and $sk^{\alpha\text{-}1}$ which means $\dot{k}/k$ < 0 (which means $\dot{y}/y$ < 0) ; it remains so until capital per worker goes back to the original level; the end result is a one time decrease output per worker (y)... this is why we care about unemployment



Problems -

- **Model is Population Driven** - growth rates of output (Y) and capital (K) equal growth rate of labor (L)
- **No Growth in Output per Worker** - model says y doesn't grow in steady state, but data for U.S. says otherwise... this is why we introduce technological change

Solow Model + Technology

Technology - captured in the production function

Hicks-Neutral - $Y = AF(K, L) = F(AK, AL)$

Capital Augmenting - $Y = F(AK, L)$

Labor Augmenting - $Y = F(K, AL)$

Results of technology are most transparent with labor augmenting technology so that's where we'll focus

Effective Labor - AL; we use A to represent the level of technology; using the labor augmenting production function, if A changes from 1 to 2, that means workers are twice as productive (we effectively get the same work as if we doubled the number of workers)

Update Model - basically have four equations that drive the model

1. $Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$
2. $A(t) = A_0 e^{gt} \;\Rightarrow\; \frac{\dot{A}}{A} = g$ (constant rate of technological advancement)... this means that technological progress is <u>exogenous</u> ("mana from heaven"; technology descends upon the economy automatically and regardless of whatever else is going on in the economy)
3. $L(t) = L_0 e^{nt} \;\Rightarrow\; \frac{\dot{L}}{L} = n$ (constant labor growth rate)... same as before
4. $\dot{K} = sY - \delta K$ ... same capital accumulation equation as before

What's Constant? - divide equation 1 by capital to get output per capital:

$$\frac{Y}{K} = \frac{K^{\alpha}(AL)^{1-\alpha}}{K} = \left(\frac{AL}{K}\right)^{1-\alpha}$$

Since both A and L grow at constant rates (g and n), K must also grow at a constant rate or else Y/K would either go to infinity or go to zero (both are unrealistic); if we invert AL/K we get capital per effective worker:

$$\tilde{k} \equiv K / AL = k / A$$

Similarly, we can define output per effective worker: $\tilde{y} \equiv Y / AL = y / A$; now we can rewrite equation 1:

$$\tilde{y} = \frac{Y}{AL} = \frac{K^{\alpha}(AL)^{1-\alpha}}{AL} = \left(\frac{K}{AL}\right)^{\alpha} = \tilde{k}^{\alpha}$$

Output per effective worker and capital per effective worker will be constant in steady state

Growth Rates - Do the $\ln$-differentiate trick on $\tilde{k}$ and $\tilde{y}$

$$\ln \tilde{k} = \ln k - \ln A \;\Rightarrow\; \frac{\dot{\tilde{k}}}{\tilde{k}} = 0 = \frac{\dot{k}}{k} - \frac{\dot{A}}{A} \;\Rightarrow\; \frac{\dot{k}}{k} = \frac{\dot{A}}{A} = g$$ $\therefore$ capital per worker (k) grows at same rate as technological progress

$$\ln\tilde{k} = \ln K - \ln A - \ln L \;\Rightarrow\; \frac{\dot{\tilde{k}}}{\tilde{k}} = 0 = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \;\Rightarrow\; \frac{\dot{K}}{K} = \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = g + n$$

$$\ln\tilde{y} = \ln y - \ln A \;\Rightarrow\; \frac{\dot{\tilde{y}}}{\tilde{y}} = 0 = \frac{\dot{y}}{y} - \frac{\dot{A}}{A} \;\Rightarrow\; \frac{\dot{y}}{y} = \frac{\dot{A}}{A} = g$$

$$\ln\tilde{y} = \ln Y - \ln A - \ln L \;\Rightarrow\; \frac{\dot{\tilde{y}}}{\tilde{y}} = 0 = \frac{\dot{Y}}{Y} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \;\Rightarrow\; \frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = g + n$$

Measure	Steady State Growth Rate
$\tilde{y}$, $\tilde{k}$	0
L	n
A , y , k	g
Y , K	$g+n$

Solving the Model - try to put capital accumulation equation in terms of $\tilde{k}$ and $\tilde{y}$

Divide by K: $\frac{\dot{K}}{K} = \frac{sY}{K} - \delta$

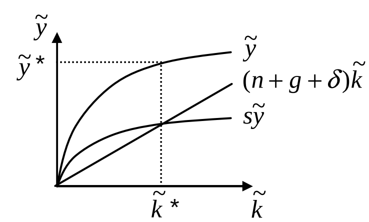
By using the $\ln$-differentiate trick on $\tilde{k}$, we found: $\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L}$

Solve that for $\frac{\dot{K}}{K}$ and substitute it into the first equation: $\frac{\dot{\tilde{k}}}{\tilde{k}} + \frac{\dot{A}}{A} + \frac{\dot{L}}{L} = \frac{sY}{K} - \delta$

Notice $\frac{\dot{A}}{A} = g$ and $\frac{\dot{L}}{L} = n$; multiply $\frac{sY}{K}$ by $\frac{AL}{AL}$: $\frac{\dot{\tilde{k}}}{\tilde{k}} + g + n = \frac{sY/AL}{K/AL} - \delta$

Use definitions $\tilde{y} \equiv Y/AL$ and $\tilde{k} \equiv K/AL$: $\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{s\tilde{y}}{\tilde{k}} - (n+g+\delta)$

Solve for $\dot{\tilde{k}}$: $\dot{\tilde{k}} = s\tilde{y} - (n+g+\delta)\tilde{k}$



Steady State - $\dot{\tilde{k}} = 0$ so $\boxed{s\tilde{y} = (n+g+\delta)\tilde{k}}$

Comparative Statics and Transitional Dynamics are similar to original model

Endogenous Growth Models

Neoclassical Model - g affects long-run growth and transition; model doesn't say anything about how g is determined; we want to find factors that determine g so we can make it endogenous (that way we can figure out how we might change g through policy)

Literature -

Neoclassical
Exogenous technical change
$A(t) = A_0e^{gt}$ (g can't be changed)
Positive population growth
$n > 0$ $L(t) = L_0e^{nt}$
Authors: Solow

Schumpeterian
Endogenous technical change
(R&D; intentional investment)

No Population Growth
$n = 0$; Scale Effects
If $n > 0$, then $\dot{y}/y$ unbounded
Authors:
Romer (1990) - variety
quality $\left\{ \begin{array}{l} \text{Segerstron, Anant, Dinopoulos (1990)} \\ \text{Aghion \& Howitt (1992)} \\ \text{Grossman \& Helpman (1995)} \end{array} \right.$

Positive Population Growth
n > 0; No Scale Effects

Exogenous *g*
Authors:
Jones (1995) - modified Romer; like neoclassical; ends up with g = αn
Segerstrom (1998) - quality

Endogenous *g*
Authors:
Young (1998)
Howitt (1998)
Dinopoulos & Thompson (1998)

Technology -

Fixed Cost - technology involves fixed costs which leads to non-convexities (economies of scale); excludes perfectly competitive markets

Suppose output provided by $Y = AL$, A = level of technology; L = amount of labor

Let $C(A)$ be the cost of technology level A

Under perfect competition, firm faces constant price P so $\pi = PY - wL - C(A)$

If labor market is perfectly competitive, wage equals the value of marginal productivity of labor: $w = PY_L = PA$

Substitute that back into the profit function: $\pi = PAL - PAL - C(A) = -C(A) < 0$

∴ Romer model focuses on imperfect competition; economic profits are required before R&D can take place

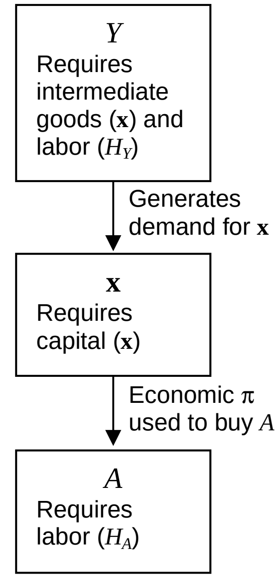
Design - technology takes form of design; instructions on how to produce a particular product

Non-Rival - if one person consumes the good, another can also consume it at the same time without diminishing the first person's benefit

Excludable - it is possible to prevent others from using the good; technology is made excludable by keeping secrets (e.g., formula for Coke), encryption (cable TV), patents/copyrights, etc.

Incentive - excludability provides incentive for market to exist because firms can charge for the product or service

	Rival	Non-Rival
Excludable	**Private (Traditional) Good** Apples Mohammad's Coke	**Collective Good** Technology Pay-per-view TV
Non-Excludable	**Commons Good** Parking Lot Fish in Ocean	**Public Good** National Defense Lighthouse



Romer Model (simplified)

3 Sectors - economy has 3 sectors that are vertically related

1. Homogeneous Good - produced under perfect competition with standard Cobb-Douglass production function
2. Intermediate Goods - produce capital goods under monopoly (imperfect competition); uses only capital goods and technology, no labor (although it can be added and we get the same result... just keeping it simple)
3. Research Sector - perfect competition prevails; only uses labor

Fixed Labor - $L = H_Y + H_A$; split determines growth rate

Basics - start with demand for Y; solve profit maximization problem; that generates demand for $\mathbf{x}$; solve monopoly profit maximization problem; that profit generates incentives in third market

Final Output - $Y = Y(H_Y, \mathbf{x}) \equiv H_Y^{\alpha}\left(\sum_{i=1}^{\infty} x_i^{1-\alpha}\right)$; we're looking at potential technologies (infinite)

Example - consider only 2 designs (i.e., A = 2): $Y = H_Y^{\alpha}\left(x_1^{1-\alpha} + x_2^{1-\alpha}\right)$

Constant x_i - each design generates the same level of profit each period (π); with an infinite time horizon, the discounted profit is π/r (regardless of when the design is discovered); $\therefore$ the optimal amount of each design is the same: $x_i = x$ (this is called **symmetry**)

Note: we'd get the same conclusion with a finite time horizon as long as each design has the same lifespan

Result - $Y = H_Y^{\alpha}\left(x^{1-\alpha} + x^{1-\alpha}\right) = 2H_Y^{\alpha}x^{1-\alpha}$ ... general form: $Y = AH_Y^{\alpha}x^{1-\alpha}$

Discontinuities - if a new design is discovered, Y jumps (i.e., is not continuous); that means we can't use calculus; to fix the problem, we'll look at a continuum of new designs:

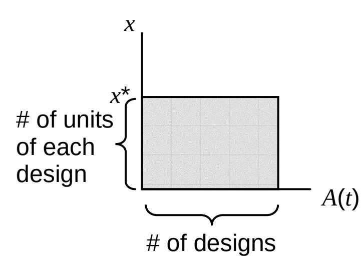
$$Y = Y(H_Y, x, A(t)) \equiv H_Y^{\alpha}\int_0^{A(t)} x(i)^{\alpha-1}di = A(t)H_Y^{\alpha}x^{1-\alpha}$$ ... because of symmetry ($x(i) = x$)

Relative Prices - $I = P_1x_1 + P_2x_2$; **Balraw's Law** - once you determine 1 price and know quantities in both markets, 2nd price is known

Numeraire - set price of 1 good to $1 so all other goods are priced relative to the numeraire (e.g., $5,000 computer and $10,000 car; if computer is numeraire, price of car is 2 [computers/car]); we'll set the final good as the numeraire

Depreciation - $\dot{K}(t) = Y(t) - C(t)$ (where $C(t)$ is aggregate consumption at time t)

Capital - by definition capital is number of intermediate goods used in production of final good so $K = \int_0^{A(t)} x(i)di = x^*\int_0^{A(t)} di = x^* A(t) \Rightarrow x = \dfrac{K}{A(t)}$



We can aggregate because all intermediate goods add same to output on margin; result of this equation is that if there is no new technology, we don't get more capital

Substitute x into the production function:

$$Y = \frac{A(t)H_Y^{\alpha}K^{1-\alpha}}{A(t)^{1-\alpha}} = (A(t)H_Y)^{\alpha}K^{1-\alpha}$$, so we have labor augmenting technological change (just like Solow model)

Empirical Data - α = 2/3... roughly 2/3 inputs into output is labor and 1/3 is capital; further breakdown shows labor equally split between skilled and unskilled

Evolution of Design - productivity defined as number of designs per period of time per researcher; this value is a constant and is dependent on the number of researchers and the number of designs that already exist

Warning - we're dealing with continuous time so intuition isn't clear

Look at j^{th} researcher; flow of designs per period of time is dA_j

That's equal to a productivity parameter (δ) times the hours worked by the j^{th} researcher (H_j) times the number of designs ($A(t)$) times the length of the time period (dt)

$$dA_j = \delta H_j A(t)dt$$

Aggregate Designs - $\sum_j dA_j = \delta A(t)dt \sum_j H_j \;\Rightarrow\; dA(t) = \delta H_A A(t)dt$, where $dA(t)$ is change in total number of designs over time and H_A is the total number of hours devoted to research

Realize that $dA(t) = \frac{\partial A(t)}{\partial t}dt = \delta H_A A(t)dt$ ... we can cancel out the dt on each side

Now divide both sides by $A(t)$ to get rate of technological change: $\frac{\partial A(t)/\partial t}{A(t)} = \frac{\dot{A}}{A} = \delta H_A$ ...

note that this rate is constant over time and depends on total hours of research

Labor - full employment constraint; total amount of labor is constant over time (no population growth) so all we can do is transfer labor between research of technology and production of final good: $H = H_Y + H_A$

Summary of Model - 5 equations:

(1) $Y = A(t)H_Y^{\alpha}x^{1-\alpha}$ — $A(t)$ = # of types of intermediate goods (# of types of computer); x = # of intermediate good (# of each type of computer)

(2) $\dot{K}(t) = Y(t) - C(t)$ — $C(t)$ = consumption

(3) $K = x * A(t)$

(4) $\frac{\dot{A}}{A} = \delta H_A$ — δ = productivity parameter

(5) $H = H_Y + H_A$

Solving the Model - we'll only focus on long-run (steady-state) equilibrium; transitional dynamics for this model are hard

Basic Idea - we have technology used for intermediate good used for final good; we'll solve them backwards

Final Good - under perfect competition; profits are $P_Y Y - wH_Y - \int_0^{A(t)} P(i)x(i)di$

We said Y was the numeraire so P_Y = 1; substitute eqn (1) so problem becomes:

$$\max_{H_Y, x(i)} \boldsymbol{\pi}_Y = H_Y^{\alpha}\int_0^{A(t)} x(i)^{1-\alpha}di - wH_Y - \int_0^{A(t)} P(i)x(i)di$$

FOC - $\frac{\partial \boldsymbol{\pi}_Y}{\partial H_Y} = 0 \;\Rightarrow\; \alpha H_Y^{\alpha-1}\int_0^{A(t)} x(i)^{1-\alpha}di = w$

To do other FOC, rewrite objective: $\int_0^{A(t)} \left[H_Y^\alpha x(i)^{1-\alpha} - P(i)x(i)\right] di - wH_Y$ and realize the integral is maximized when the term in brackets is maximized so the FOC is $\frac{\partial \boldsymbol{\pi}_Y}{\partial x(i)} = 0 \;\Rightarrow\; (1-\alpha)H_Y^\alpha x(i)^{-\alpha} = P(i)$ ... this is **inverse demand** for $x(i)$

Easier Way - use the symmetry before taking FOC:

$$\max_{H_Y, x(i)} A(t)H_Y^\alpha x^{1-\alpha} - wH_Y - \int_0^{A(t)} P(i)x(i)di$$

FOC - $\alpha A(t)H_Y^{\alpha-1}x^{1-\alpha} = w$ and $(1-\alpha)H_Y^\alpha x^{-\alpha} = P$

Intermediate Goods - each good has same inverse demand $P(x) = (1-\alpha)H_Y^\alpha x^{-\alpha}$ ... because of symmetry discussed early, we dropped the i subscript and only talk about x as the number of each design (we use the same number of each design); intermediate goods are produced by monopolists ∴

$$\max_x \boldsymbol{\pi}_x = P(x)x - rx = (1-\alpha)H_Y^\alpha x^{1-\alpha} - rx$$

Wage of Capital - interest rate (r) is the wage (cost) of using capital (x) because it's the **opportunity cost** (value of best alternative... could use the intermediate good or invest it and get a return of r)

FOC - $\frac{\partial \boldsymbol{\pi}_x}{\partial x} = 0 \;\Rightarrow\; (1-\alpha)\left[(1-\alpha)H_Y^\alpha x^{-\alpha}\right] = r \;\Rightarrow\; (1-\alpha)P(x) = r \;\Rightarrow\; P(x) = \frac{r}{1-\alpha}$, so price is a fraction of r (which is constant in steady-state)

Max Profit - $\boldsymbol{\pi}_x = P(x)x - rx = P(x)x - (1-\alpha)P(x)x = \alpha P(x)x > 0$, so profit is positive and a fraction of total sales (constant in steady-state)

Technology - "market for designs"; look at value of the firm; let P_A = price of a design; there are 2 ways to solve for P_A:

Stock Market Arbitrage - rate of return of bonds must equal the rate of return of "stocks" (technology)

Bonds - return $r(t)dt$ (using continuous time)

Stocks - return is based on dividend + capital gain

Dividend - return for purchasing technology is monopoly profit of intermediate good so divide that profit by P_A : $\frac{\boldsymbol{\pi}_x(t)}{P_A}dt$

Capital Gains - change in price of technology over price paid: $\frac{dP_A}{P_A}$

Putting it together: $\frac{\boldsymbol{\pi}_x(t)}{P_A}dt + \frac{dP_A}{P_A} = r(t)dt$

Substitute: $dP_A = \frac{\partial P_A}{\partial t}dt = \dot{P}_A dt \;\Rightarrow\; \frac{\boldsymbol{\pi}_x(t)}{P_A}dt + \frac{\dot{P}_A}{P_A}dt = r(t)dt$

Cancel the dt: $\boxed{\frac{\boldsymbol{\pi}_x(t)}{P_A} + \frac{\dot{P}_A}{P_A} = r(t)}$ (Stock Market Arbitrage Equation; <u>always holds</u>)

Empirical Data - tested with S&P 500; holds within 0.29 over last 20 years

Discounted Profit Stream - based on price of design being equal to present value of all future profits; we'll use non-constant interest rate so instead of e^{-rt} we'll have an integral... this is the way Romer did it

$$P_A = \int_t^{\infty} e^{-\int_t^{\tau} r(s)ds} \boldsymbol{\pi}(\tau)d\tau$$

Leibnitz Rule (τ not function of t so only 2 terms)

Differentiate wrt t: $\dfrac{dP_A}{dt} = \dot{P}_A = -\dfrac{dt}{dt} e^{-\int_t^{\tau} r(s)ds} \boldsymbol{\pi}(\tau)\Big|_t + \int_t^{\infty} \dfrac{d}{dt}\left[e^{-\int_t^{\tau} r(s)ds} \boldsymbol{\pi}(\tau) \right] d\tau$

Look at first term: $\dfrac{dt}{dt} = 1$ and evaluate at t means $\tau = t$ so

$$-\frac{dt}{dt} e^{-\int_t^{\tau} r(s)ds} \boldsymbol{\pi}(\tau)\Big|_t = -e^{-\int_t^{t} r(s)ds} \boldsymbol{\pi}(t)$$

Since $\int_t^t r(s)ds = 0$, the first term boils down to $-e^0 \boldsymbol{\pi}(t) = -\boldsymbol{\pi}(t)$

Look at second term:

$\boldsymbol{\pi}(\tau)$ isn't a function of t so we can pull it out of the derivative

$\dfrac{d}{dt} e^{f(t)} = f'(t)e^{f(t)}$; in this case $f(t) = -\int_t^{\tau} r(s)ds$

Use Leibnitz Rule again (only 1 term): $f'(t) = \dfrac{dt}{dt} r(s)\Big|_t = r(t)$

So right term becomes: $\int_t^{\infty} \dfrac{d}{dt}\left[e^{-\int_t^{\tau} r(s)ds} \boldsymbol{\pi}(\tau) \right] d\tau = \int_t^{\infty} r(t) e^{-\int_t^{\tau} r(s)ds} \boldsymbol{\pi}(\tau)d\tau$

Now pull out the $r(t)$: $r(t)\int_t^{\infty} e^{-\int_t^{\tau} r(s)ds} \boldsymbol{\pi}(\tau)d\tau = r(t)P_A$

Combine terms: $\dot{P}_A = -\boldsymbol{\pi}(t) + r(t)P_A \;\Rightarrow\; \dfrac{\boldsymbol{\pi}_x(t)}{P_A} + \dfrac{\dot{P}_A}{P_A} = r(t)$ (same equation)

Key Difference - this method gets to stock market arbitrage equation, but can't go the other way because of integration constant; basically we'd have $P_A + c$ which explains how we can have "bubbles" in market; stock market arbitrage equation still holds because constant drops out when we differentiate by t, but stock price is actually higher (or lower) than P_A

Value of Firm - solve stock market arbitrage equation for P_A: $P_A = \dfrac{\boldsymbol{\pi}_x(t)}{r(t) - \dot{P}_A / P_A}$; that's instantaneous profit divided by instantaneous interest rates minus growth rate of firm

Steady-State - both methods gave us the stock market arbitrage; under stead-state:

$$\frac{\dot{P}_A}{P_A} = 0 \;\Rightarrow\; P_A = \frac{\pi_x(t)}{r(t)}$$

Researcher Wage - productivity per worker times value of output: $\delta A(t)P_A$

Productivity Per Worker - productivity parameter times number of designs: $\delta A(t)$

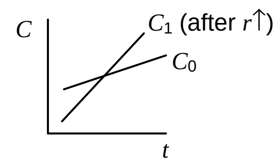
Value of Output - we just found that: P_A

Consumer Behavior - studied producer side for final good, intermediate goods, and technology; last piece of the puzzle is consumer side; assume utility is discounted at constant rate ρ:

$$\max \int_0^\infty \left[\frac{C^{1-\sigma}-1}{1-\sigma}\right] e^{-\rho t} dt \text{ s.t. } \dot{Z}(t) = rZ(t) + w - C(t)$$

Constraint - change in assets = interest from assets plus wage minus consumption

dynamic optimization... maybe next time



Solution - $\dfrac{\dot{C}}{C} = \dfrac{r(t)-\rho}{\sigma}$

Slope of consumption (says nothing about <u>level</u>); note, if $r\uparrow$, slope is steeper (consumers will [eventually] consume more in the future)

Summary of Solution so Far -

(1) $w = \alpha H_Y^{\alpha-1} \int_0^{A(t)} x(i)^{1-\alpha} di = \alpha A(t) H_Y^{\alpha-1} x^{1-\alpha}$ from solving max $\boldsymbol{\pi}_Y$

(2) $P(x) = (1-\alpha) H_Y^{\alpha} x^{-\alpha}$ from solving max $\boldsymbol{\pi}_Y$

(3) $P(x) = \dfrac{r}{1-\alpha}$ from solving max $\boldsymbol{\pi}_x$

(4) $\boldsymbol{\pi}_x = \alpha P(x) x$ from solving max $\boldsymbol{\pi}_x$

(5) $P_A = \dfrac{\pi_x(t)}{r(t)}$ from solving stock market arbitrage eqn

(6) $w_R = \delta A(t) P_A$ researcher wage

(7) $\dfrac{\dot{C}}{C} = \dfrac{r(t)-\rho}{\sigma}$ from consumer behavior

Steady-State Solution - all 7 above hold, but we have steady state number each design (intermediate goods) so we'll use x^*; also that relates to a single price so $P(x^*) = P^*$

Labor Market Equilibrium - wage in manufacturing of final good (Y) is same as wage of researchers (so there's no movement of workers form one to the other):

$$w = \alpha A(t) H_Y^{\alpha-1} (x^*)^{1-\alpha} = \delta A(t) P_A$$

Take (5) and substitute (4): $P_A = \dfrac{\alpha P^* x^*}{r(t)}$

Substitute (2) into that: $P_A = \dfrac{\alpha \left[(1-\alpha) H_Y^{\alpha} (x^*)^{-\alpha}\right] x^*}{r(t)}$

Plug that into the right side of the wage equilibrium equation:

$$w = \alpha A(t) H_Y^{\alpha-1} (x^*)^{1-\alpha} = \delta A(t) \frac{\alpha \left[(1-\alpha) H_Y^{\alpha} (x^*)^{-\alpha}\right] x^*}{r(t)}$$

A bunch of stuff cancels out: $H_Y^{-1} = \dfrac{\delta(1-\alpha)}{r(t)} \Rightarrow H_Y = \dfrac{r(t)}{\delta(1-\alpha)}$

4 Equations & 4 Unknowns - solve for H_A, H_Y, g, r

[1] $H_Y = \dfrac{r(t)}{\delta(1-\alpha)}$ from solving labor market equilibrium

[2] $g = \dfrac{\dot{A}}{A} = \delta H_A$ solved when talking about evolution of design

[3] $H = H_A + H_Y$ fixed labor

[4] $g = \dfrac{\dot{C}}{C} = \dfrac{r(t) - \rho}{\sigma}$ first part is assumption; second part from consumer behavior

Solve [3] for H_A and plug it into [2]: $g = \delta(H - H_Y)$

Substitute H_Y from [1] into this: $g = \delta\left(H - \dfrac{r(t)}{\delta(1-\alpha)}\right) = \delta H - \dfrac{r(t)}{1-\alpha}$

Solve [4] for r and plug it into this: $g = \delta H - \dfrac{g\sigma + \rho}{1-\alpha}$

Solve for g: $g = \dfrac{\delta H - \dfrac{\rho}{1-\alpha}}{1 + \dfrac{\sigma}{1-\alpha}} = \dfrac{(1-\alpha)\delta H - \rho}{1 - \alpha + \sigma}$

(Dinopoulos did left version in class; I got second version... they're the same)

Policy - what can we do to target g?

Subsidize R&D - $\delta\uparrow \Rightarrow g\uparrow$

Less Consumption Now - $\rho\downarrow \Rightarrow g\uparrow$... not sure you can target patience with policy, but we can look at societies with higher savings rates (less consumption) and find higher g

Growth Rates - turns out everything grows at g... $g = \dfrac{\dot{A}}{A} = \dfrac{\dot{C}}{C} = \dfrac{\dot{Y}}{Y} = \dfrac{\dot{K}}{K}$

$Y = H_Y^{\alpha} x^{1-\alpha} A(t)$ ... H_Y^{α} and $x^{1-\alpha}$ fixed so ln-differentiate trick gives $\dfrac{\dot{Y}}{Y} = \dfrac{\dot{A}}{A} = g$

$K = A(t)x$ ... x fixed so ln-differentiate trick gives $\dfrac{\dot{K}}{K} = \dfrac{\dot{A}}{A} = g$

Problem - H in numerator suggests larger H yields larger g; that would suggest U.S. growth is faster than Hong Kong's (it's not!); also H grows exponentially so g has to grow exponentially (i.e., t $\to \infty \Rightarrow g \to \infty$)... that's not realistic either

Schumpeterian Growth - using quality for technological progress

Papers -

Segerstrom, et.al. - AER, 1990
Aghion & Howitt - intermediate goods (like Romer); Econometrica, 1992
Grossman & Helpman - continuum of industries; 1991
Dinopoulos - Overview; 1993

Characteristics -

- **Dynamic General Equilibrium Model** - can't use partial equilibrium because 1 market growing could draw resources away from other markets
- **Product Replacement** - "creative destruction"; products are replaced by "better" products (e.g., typewriter, VHS tapes, carburetors)
  - **Imperfect Competition** - development of new products requires at least temporary monopoly to do R&D
- **Uncertainty** - characterizes all new product development... only about 10% survive

Structure of Model - "all the difficult parts of economics come together"

- **One Good** - one industry producing a single consumption good
- **Quality** - quality of good can be improved; all "versions" of final good are perfect substitutes
- **R&D Races** - uncertainty
- **Two Activities** - manufacturing of final output and R&D for quality improvement
- **Full Employment** - fixed labor force
- **Monopoly** - firm that wins R&D race enjoys monopoly; duration of monopoly depends on next R&D race
- **Price Limit** - price limited by price of previous good and amount of improvement
- **Stock Market** - used to finance R&D

Utility - representative consumer has intertemporal utility function: $U = \int_0^\infty e^{-\rho t} \ln[z(\bullet)]dt$

- **Discount rate** - ρ
- **Sub-utility** - $z(x_0, x_1, x_2, \ldots) = \sum_{q=0}^{\infty} \alpha^q x_q = x_0 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \cdots$
  - **Degree of Improvement** - α > 1 is degree of quality improvement of product relative to its immediate predecessor
  - **Version of Good** - x_q; countably infinite levels of quality

Product Replacement Mechanism - sub-utility above works (in conjunction with pricing) to effectively eliminate previous versions of the final good

- **Example** - suppose each worker produces 1 unit of output (x) and use wage of labor as numeraire; that means 1 worker = 1 unit = $1... MC = AC = 1

  Now suppose economy starts with only x_0; consumers spend all their money on this good so demand is $x_0 = \frac{E}{P_0}$, where E is aggregate expenditures, P_0 is price of x_0

  Suppose x_1 is discovered; consumer utility is $x_0 + \alpha x_1$; if x_0 and x_1 are at the same price, consumer buys all x_1 and no x_0; firm that discovered x_1 wants to drive producer of x_0 out of market, but also wants to make as much profit as possible so he charges the highest price he can for x_1 that still has consumers choosing only x_1... i.e., want to keep utility form x_1 higher than utility from x_0: $\alpha x_1 \geq x_0$; if consumers spend all their money on either good we have $x_0 = \dfrac{E}{P_0}$ and $x_1 = \dfrac{E}{P_1}$;

substitute these into the utility restriction and solve for P_1:

$$\alpha \frac{E}{P_1} \geq \frac{E}{P_0} \;\Rightarrow\; \alpha P_0 \geq P_1$$

Since minimum price of producer of x_0 is MC = wage (w), then the limit price of x_1 is $P_q \leq \alpha P_{q-1}$; if $\alpha w = P_1$, consumers are indifferent between x_0 and x_1 but assume hey switch instantaneously (i.e., always choose higher quality when indifferent)

In General - start with utility: $\alpha^{q-1} \dfrac{E}{P_{q-1}} \leq \alpha^q \dfrac{E}{P_q} \;\Rightarrow\; P_q \leq \alpha P_{q-1} \;\Rightarrow\; P_q \leq \alpha P_{q-1}$

Profits of Monopolists - assume one worker manufactures one unit of good x_q independent of the level of quality (i.e., doesn't matter what level of quality is, only takes 1 worker)

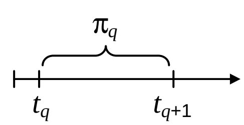
Demand for x_q = $\begin{cases} \dfrac{E}{P_q} & \text{if } P_q \leq \alpha w \\ 0 & \text{if } P_q > \alpha w \end{cases}$ (we substitute w (marginal cost) for P_{q-1})

Firm Objective - max $\pi = (P_q - w)x_q = (w\alpha - w)\dfrac{E}{w\alpha} = \boxed{\dfrac{\alpha - 1}{\alpha}E}$ (i.e., profits are proportional to consumptions (like the Romer model)

Solution - to maximize profit, firm charges maximum price so $\boxed{P_q = \alpha w}$

Model Arrival of Innovations -

Poisson Process - characterized by intensity μ ("velocity of innovations")



# of events x that will take place over any interval of length Δ ~ Poisson

$$g(x) = \Pr[\,x \text{ events occur}\,] = \frac{(\mu\Delta)^x e^{\mu\Delta}}{x!}$$

Time T you will have to wait for X to occur ~ Exponential

$F(T) \equiv \Pr[\text{event occurs before } T] = 1 - e^{-\mu T}$ (cdf)

$f(T) = F'(T) = \mu e^{-\mu T}$ (pdf)

$\therefore$ probability that event will occur sometime within the short interval between T and T + dt is approximately $\mu e^{-\mu T} dt$

Instantaneous Probability - $\lim_{T \to 0} \mu e^{-\mu T} dt = \mu dt$

instantaneous probability that event doesn't occur is $1 - \mu dt$

Expected Time Between Intervals - 1 / μ

Independent Firms - add intensity levels; instantaneous probability is $(\mu_1 + \mu_2)\,dt$; probability that firms discover innovation at same time is zero

Aggregate Labor - labor used for R&D is L; firm j's labor is L_j so $L = \sum_j L_j$

Diminishing Returns - measured by $0 < \gamma \leq 1$... we'll use $\gamma = 1$ to keep things simple

Combine it all: instantaneous probability that at least one firm will discover new product at time dt is $\mu(L)dt = \left(\sum_j \mu_j\right)dt = L^\gamma dt$

"Fair" Research - assume firm j's relative instantaneous probability of success equals its share of R&D resources: $\frac{\mu_j}{\mu} = \frac{L_j}{L}$

Probability of Success - individual firm's instantaneous probability of discovering next higher quality good is $\mu_j dt$

Solve "fair" research assumption for μ_j and sub into this probability: $\mu_j dt = \mu \frac{L_j}{L} dt$

Sub for μ (i.e., $\mu(L)$) from above: $\mu_j dt = (L^\gamma)\frac{L_j}{L}dt = \boxed{L_j L^{\gamma-1} dt}$

Firm Doing R&D - denote discounted earnings of winner of R&D race as $V(t)$

Expected Discounted Earnings - have to multiply by probability of winning: $V(t)L_j L^{\gamma-1}dt$

Cost for R&D - $wL_j dt$

Expected Discounted Profits - $V(t)L_j L^{\gamma-1}dt - wL_j dt$

Free Entry - if we assume free entry into R&D Race, expected profits will be zero $\therefore$ $V(t)L_j L^{\gamma-1}dt - wL_j dt = 0 \;\; \Rightarrow \;\; V(t) = wL^{1-\gamma}$ (or if we let $\gamma = 1$, $V(t) = w$); this is the **stock market value of the firm**

Financing R&D - firms issues stock: "If I win R&D race by time dt, stockholders get monopoly profit until next R&D race; if I don't win, stockholders get zero"; very risky stock

2 Types of Firms - (1) monopolist producing x_q; (2) R&D firms looking for x_{q+1}

Risk Free Return - $r(t)$; return on risk free bond in time dt is $r(t)dt$

Return for Stock - of firm that has monopoly on current good;

$$\frac{\pi}{V(t)}dt + \frac{dV}{V(t)}(1 - L^\gamma dt) - \frac{V(t) - 0}{V(t)}L^\gamma dt$$

dividends + capital gains * Pr(firm survives) - value of firm * Pr(firm disappears)

Note: if L = 0 this is the same as the stock market formula use in Romer Model

Trick - $dV = \frac{\partial V}{\partial t}dt = \dot{V}dt$

Portfolio Efficiency - expected return of security of existing monopolist must be equal to the risk free rate of return:

$$\frac{\pi}{V(t)}dt + \frac{\dot{V}dt}{V(t)}(1 - L^\gamma dt) - L^\gamma dt = r(t)dt$$

dt's cancel: $\frac{\pi}{V(t)} + \frac{\dot{V}}{V(t)}(1 - L^\gamma dt) = r(t) + L^\gamma$

$\lim_{dt \to 0}$: $\frac{\pi}{V(t)} + \frac{\dot{V}}{V(t)} = r(t) + L^\gamma$

Solve for $V(t)$: $V(t) = \dfrac{\boldsymbol{\pi}}{r(t) + L^{\gamma} - \dfrac{\dot{V}}{V(t)}}$

Empirical Support - current research shows that this equation works for the S&P 500; doesn't seem to work for individual firms because L is difficult to estimate)

Labor - 1 worker makes 1 unit of output; output is $x = \dfrac{E}{\alpha w} = \dfrac{E}{\alpha}$ (since we're assuming w = 1) $\therefore$ number of workers in manufacturing is E/α

Confusion over L -

<u>Labor used for R&D</u> is L - this is the original definition

<u>Intensity of R&D</u> (Poisson process) is $\mu(L) = L^{\gamma}$ - we assumed $\gamma = 1$ so intensity is L

<u>Instantaneous probability that new discovery</u> is made during interval dt is $\mu(L)dt = L^{\gamma}dt$ - we assumed $\gamma = 1$ so instantaneous probability is Ldt

For a firm j, this probability is $\mu_j dt = L_j L^{\gamma-1} dt$

<u>Risk of default</u> is probability of new development in dt which we just said was Ldt if $\gamma = 1$

Consumer Maximization - $\max\limits_{E}\ U = \int_0^{\infty} e^{-\rho t} \ln[z(\bullet)]dt$ s.t. $\dot{A}(t) = rA + w - E$

(solve in HW)

Solution - $\dfrac{\dot{E}}{E} = r(t) - \rho$

Summary of Model - 6 equations:

(1) $\boldsymbol{\pi} = (P_q - w)x_q = (w\alpha - w)\dfrac{E}{w\alpha} = \dfrac{\alpha - 1}{\alpha}E$ (monopoly profits; w = 1)

(2) $P_q = \alpha w$ (maximizes monopoly profits; w = 1)

(3) $V(t) = wL^{1-\gamma} = 1$ (free entry to R&D or zero profit condition; w = 1 and γ= 1)

(4) $V(t) = \dfrac{\boldsymbol{\pi}}{r(t) + L^{\gamma} - \dfrac{\dot{V}}{V(t)}}$ (stock market portfolio efficiency condition; γ= 1)

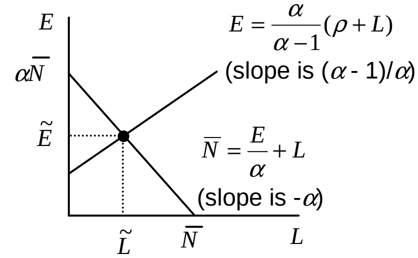
(5) $\overline{N} = \dfrac{E}{\alpha} + L$ (full employment)

(6) $\dfrac{\dot{E}}{E} = r(t) - \rho$ (consumer utility maximization)

Steady-State Solution - all 6 equations hold all the time, but in steady state we know that (6) = 0 so that means $r(t) = \rho$

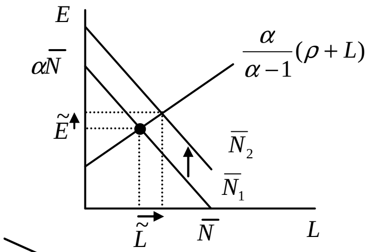
From (3) we have $V(t) = 1$ so that means $\dot{V}(t) = 0$ $\therefore$ from (4) $V(t) = \dfrac{\boldsymbol{\pi}}{r(t) + L} = 1$

Sub (1) and $r(t) = \rho$: $V(t) = \dfrac{\frac{\alpha - 1}{\alpha} E}{\rho + L} = 1$



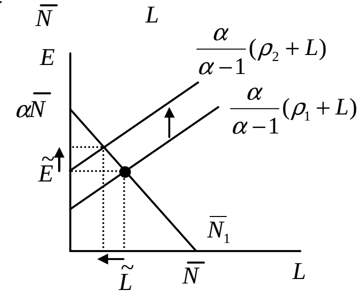
Solve for E: $E = \dfrac{\alpha}{\alpha - 1}(\rho + L)$ ... line in (E,L) space (i.e., consumption vs. investment) with constant slope

(5) gives another line in (E,L) space



Effect of Size of Economy - $\bar{N} \uparrow \Rightarrow \tilde{E} \uparrow$ and $\tilde{L} \uparrow$; more population implies more R&D investment so growth increases; this is problem with model so we'll remove the scale effects later

Consumer Patience - $\rho \uparrow \Rightarrow \tilde{E} \uparrow$ and $\tilde{L} \downarrow$; more impatient (higher discounting for future consumption) means consume more, but invest less



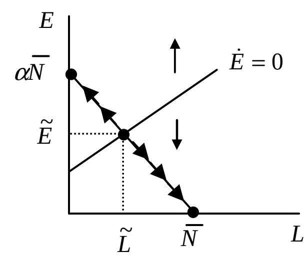
Transitional Dynamics - basically do the same thing we did for steady state, but we can't assume (6) = 0

From (3) we have $V(t) = 1$ so that means $\dot{V}(t) = 0$

Sub into (4): $V(t) = \dfrac{\boldsymbol{\pi}}{r(t) + L} = 1$

Solve for profit: $\boldsymbol{\pi} = r(t) + L$

That equals (1): $\boldsymbol{\pi} = r(t) + L = \dfrac{\alpha - 1}{\alpha} E$



Solve for interest rate: $r(t) = \dfrac{\alpha - 1}{\alpha} E - L$

Sub into (6): $\dfrac{\dot{E}}{E} = \dfrac{\alpha - 1}{\alpha} E - L - \rho$

Phase Diagram - look at $\dot{E} = 0$; to determine direction of movement, hold L constant and use E + dE (dE > 0);

$\dfrac{\dot{E}}{E} = \dfrac{\alpha - 1}{\alpha}(E + dE) - L - \rho > 0$ ... that means E increases above $\dot{E} = 0$ and decreases below it

There are two stable points on the graph, but neither is a feasible equilibrium

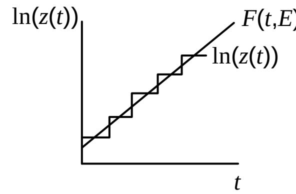
All Consumption - all consumption and no investment is not rational; firm has incentive to do R&D and have monopoly power forever... not realistic

All Investment - the other stable point on the graph doesn't maximize utility because there's no consumption

Steady-State - the steady-state point is an equilibrium, but it is unstable; ∴ transitions are jumps in E and L (realistic because E is chosen by consumers and L by firms and both choices are made instantaneously)

Growth Rate - of utility: $U = \int_0^\infty e^{-\rho t} \ln[z(\bullet)] dt$ ... really just need to focus on instantaneous utility

$\ln[z(t)] = \ln[\alpha^q x] = q \ln \alpha + \ln x$



Substitute $x = \dfrac{E}{\alpha w} = \dfrac{E}{\alpha}$: $\ln[z(t)] = q\ln\alpha + \ln E - \ln\alpha$

$F(t,E) \equiv$ expected value of instantaneous utility $= E(q)\ln\alpha + \ln E - \ln\alpha$

q is number of innovations so (a) it jumps incrementally, (b) is governed by a Poisson process so $E(q) = Lt$ (expected number of innovations from now until time t is Lt

Growth Rate = $\boxed{\dfrac{\partial F}{\partial t} = L\ln\alpha}$ $\therefore$ L (# workers in R&D)↑ ⇒ growth rate of utility↑

Welfare Analysis - sub $F(t,E)$ into $U = \int_0^\infty e^{-\rho t}\ln[z(\bullet)]dt$ to compare socially optimal level of investment and consumption to the levels determined by market equilibrium

$$\max_{E,L} U = \int_0^\infty e^{-\rho t}[tL\ln\alpha + \ln E - \ln\alpha]dt \text{ s.t. } \overline{N} = \frac{E}{\alpha} + L$$

Solve integral first:

$$U = \int_0^\infty e^{-\rho t}[tL\ln\alpha + \ln E - \ln\alpha]dt = \int_0^\infty e^{-\rho t}[tL\ln\alpha]dt + \int_0^\infty e^{-\rho t}[\ln E]dt - \int_0^\infty e^{-\rho t}[\ln\alpha]dt$$

Last two integrals are easy because $\ln E$ and $\ln\alpha$ are constant (wrt t):

$$\int_0^\infty e^{-\rho t}[\ln E]dt = \frac{\ln E}{-\rho}e^{-\rho t}\Big]_0^\infty = \frac{\ln E}{\rho}$$

$$\int_0^\infty e^{-\rho t}[\ln\alpha]dt = \frac{\ln\alpha}{-\rho}e^{-\rho t}\Big]_0^\infty = \frac{\ln\alpha}{\rho}$$

First integral needs integration by parts:

$$\int_a^b XdY = XY\Big]_a^b - \int_a^b YdX \text{ ... let } X = t \text{ and } dY = e^{-\rho t} \Rightarrow X' = dt \text{ and } Y = e^{-\rho t}/-\rho$$

$$L\ln\alpha\left[\int_0^\infty e^{-\rho t}[t]dt\right] = L\ln\alpha\left\{\frac{te^{-\rho t}}{-\rho}\Big]_0^\infty - \int_0^\infty \frac{e^{-\rho t}}{-\rho}dt\right\} = L\ln\alpha\left\{\left(\frac{\infty e^{-\rho\infty}}{-\rho} - \frac{0e^{-\rho 0}}{-\rho}\right) - \frac{e^{-\rho t}}{\rho^2}\Big]_0^\infty\right\} =$$

$$L\ln\alpha\left\{-\frac{e^{-\rho\infty}}{\rho^2} + \frac{e^{-\rho 0}}{\rho^2}\right\} = L\ln\alpha\left\{\frac{1}{\rho^2}\right\} = \frac{L\ln\alpha}{\rho^2}$$

All future growth from investment

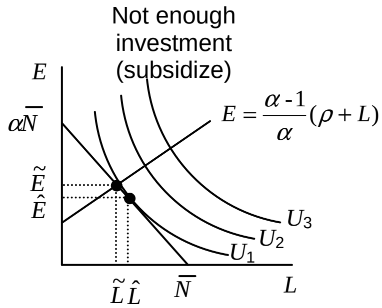
$$\text{Put it all together: } U = \frac{\ln E}{\rho} - \frac{\ln\alpha}{\rho} + \frac{L\ln\alpha}{\rho^2} = \frac{1}{\rho}\left(\underbrace{\ln E - \ln\alpha}_{\text{Real consumption expenditure}} + \underbrace{\frac{L\ln\alpha}{\rho}}_{\text{Investment}}\right)$$

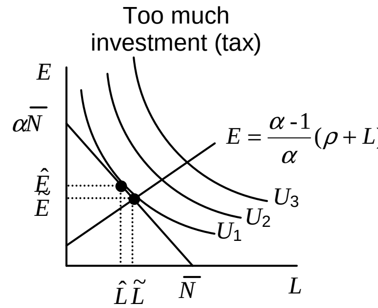
Discounted

Now it's a static problem: $\max_{E,L} U = \dfrac{1}{\rho}\left(\ln E - \ln\alpha + \dfrac{L\ln\alpha}{\rho}\right)$ s.t. $\overline{N} = \dfrac{E}{\alpha} + L$

This is a classic consumer optimization problem (standard well, behaved indifference curves and a linear budget constraint); let solution (social optimum) be $(\hat{L},\hat{E})$

Distortion - don't have enough information to give a general statement about the social welfare, but the graphs show that it's possible to have too much investment (too little consumption) or too little investment (too much consumption); usually the latter





Scale Effects and Schumpeterian Growth

Structure of Growth Models

Labor in Manufacturing - L_Y

Labor in Research - L_A

Output - $Y(t) = A(t)L_Y(t)$ (1)

Scale Effect - $X(t)$ is related to scale effects; $X(t)\uparrow \Rightarrow$ R&D more difficult

Growth of Technology - $g_A = \dfrac{\dot{A}(t)}{A(t)} = \dfrac{L_A(t)}{X(t)}$ (2)

Labor Market - $L(t) = L_Y(t) + L_A(t)$ (3)

Growth of Labor (Population) - $L(t) = L_0 e^{g_N t} \;\; \Rightarrow \;\; g_L = \dfrac{\dot{L}(t)}{L(t)} \geq 0$ (constant) (4)

Share of Labor -

Manufacturing Share - $S(t) = \dfrac{L_Y(t)}{L(t)} \;\; \Rightarrow$ **Research Share** - $1 - S(t) = \dfrac{L_A(t)}{L(t)}$

Growth Rate of Per Capital Output - write $Y(t)$ using share of labor: $Y(t) = A(t)S(t)L(t)$

Divide by $L(t)$: $y = \dfrac{Y(t)}{L(t)} = A(t)S(t)$

Do $\ln$-differentiate trick: $\ln y = \ln A(t) + \ln S(t) \;\; \Rightarrow \;\; g_y = \dfrac{\dot{y}}{y} = \dfrac{\dot{A}}{A} = g_A$

(Share of labor in both sectors must be constant in long run)

Notation Comment - common to omit (t) argument for variables that are constant in steady state

Scale Effects -

Sub research share into (2): $\boxed{g_y = \dfrac{(1-S)L(t)}{X(t)}}$ [1]

Divide (3) by $L(t)$: $\dfrac{L(t)}{L(t)} = \dfrac{L_Y(t)}{L(t)} + \dfrac{L_A(t)}{L(t)} = 1$

Sub $L_A(t) = g_y X(t)$ from (2): $\boxed{S + \dfrac{X(t)g_y}{L(t)} = 1}$ [2]

Any specification of $X(t)$ must be consistent with these two equations

Constant $X(t)$ - Assume $X(t) = x_0$ (constant)

[1] becomes $g_y = \dfrac{(1-S)L(t)}{x_0}$... which says output grows exponentially (at same rate as labor force)

[2] becomes $S+\frac{x_0 g_y}{L(t)}=1$... which says $S \underset{t\to\infty}{\to} 1$ (i.e., all workers to manufacturing) so g_y goes to zero... contradiction

Romer & Early Schumpeterian Models - solved problem of scale effects by making L and X constant (no population growth)... not realistic ("not empirically relevant)

Solution - let L and X grow at same rate

Jones - $X(t)=[A(t)]^{1/\phi}$, $\phi>0$ ... basically says as you have more designs, there are diminishing returns

Check [1]: $g_y=\frac{(1-S)L(t)}{[A(t)]^{1/\phi}}$

Do $\ln$-differentiate trick: $\ln g_y=\ln L(t)-\frac{1}{\phi}\ln A(t) \;\Rightarrow\; \frac{\dot{g}_y}{g_y}=\frac{\dot{L}}{L}-\frac{1}{\phi}\frac{\dot{A}}{A}=g_L-\frac{1}{\phi}g_y$

Don't want g_y to change in steady-stage so $g_N-\frac{1}{\phi}g_y=0 \;\Rightarrow\; g_y=\phi g_L$

Result - growth rate is exogenous (based on growth of labor)... same as Solow model

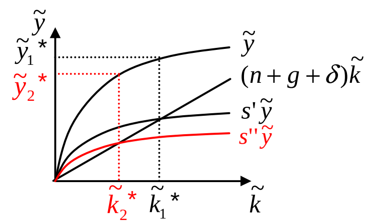
Dinopoulos, et.al. - $X(t)=\beta L(t)$ (vertical and horizontal differentiation)

Check [1]: $g_y=\frac{(1-S)L(t)}{\beta L(t)}=\frac{(1-S)}{\beta}$... $\therefore$ g_y constant, but can be changed if we change the share of workers in R&D... realistic

Check [2]: $S+\frac{\beta L(t)g_y}{L(t)}=S+\beta g_y=1 \;\Rightarrow\; g_y=\frac{1-S}{\beta}$... consistent with [1]

Quality Growth - $X(t)=$ resources to protect monopoly profit of incumbent firm (makes R&D harder

1. *A decrease in the investment rate*. Suppose the U.S. Congress enacts legislation that discourages saving and investment, such as the elimination of the investment tax credit that occurred in 1990. As a result, suppose the investment rate falls permanently from s' to s''. Examine the policy change in the Solow model with technological progress, assuming that the economy begins in steady state. Sketch a graph of how (the natural log of) output per worker evolves over time with and without the policy change. Make a similar graph for the growth rate of output per worker. Does the policy change permanently reduce the *level* or the *growth rate* of output per worker?



Concept. Don't think in terms of time; think of the destination (new steady state) and then worry about how you get there.

Steady State. Decrease in s causes steady state output per *effective* worker and capital per *effective* worker to decline. Based on the formulas:

$$y = A\tilde{k}^{\alpha} \text{ and } k = \tilde{k}/A$$

steady state output per worker and capital per worker will also decline. The growth rates, however will not be affected in the steady state:

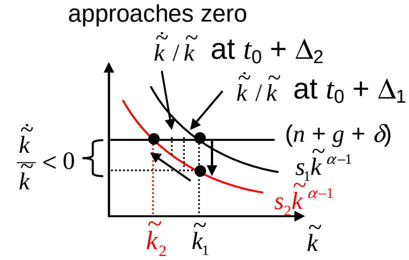
$$\frac{\dot{\tilde{y}}}{\tilde{y}} = \frac{\dot{\tilde{k}}}{\tilde{k}} = 0 \quad \text{and} \quad \frac{\dot{y}}{y} = \frac{\dot{k}}{k} = g$$

Transition. Focus on two equations that are valid all the time (not just in steady state):

$$\tilde{y} = \tilde{k}^{\alpha} \quad \text{and} \quad \dot{\tilde{k}} = s\tilde{y} + (n+g+\delta)\tilde{k}$$

The second equation can be rewritten by substituting $\tilde{y} = \tilde{k}^{\alpha}$ and dividing both sides by $\tilde{k}$:

$$\frac{\dot{\tilde{k}}}{\tilde{k}} = \frac{s}{\tilde{k}^{1-\alpha}} - (n+g+\delta)$$



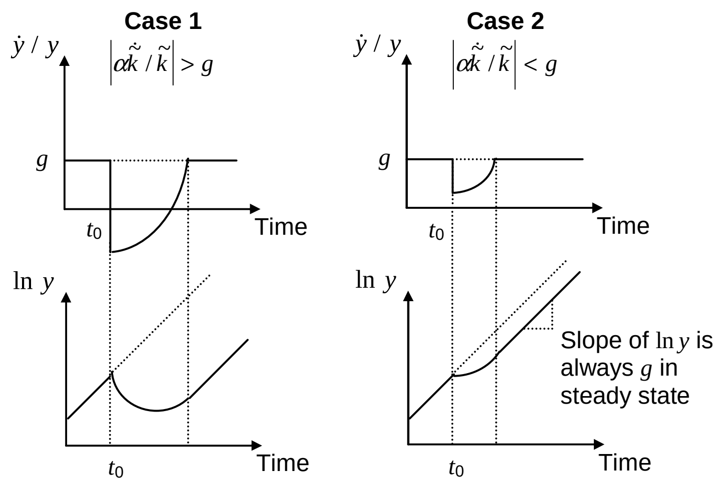
This helps us describe the transitional dynamics by plotting $s\tilde{k}^{\alpha-1}$ and $(n+g+\delta)$. The difference between the curves gives $\dot{\tilde{k}}/\tilde{k}$, the growth rate of capital per effective worker; in this case, the lower savings rate drops the $s\tilde{k}^{\alpha-1}$ curve so $\dot{\tilde{k}}/\tilde{k} < 0$. To determine the effect on output per worker, we start with the modified production function that relates $\tilde{k}$ to y:

$$y = A\tilde{k}^{\alpha}$$

To figure out what happens to y, we do the $\ln$-differentiate trick:

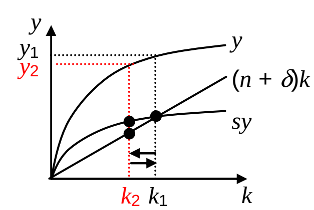
$$\frac{\dot{y}}{y}=\frac{\dot{A}}{A}+\alpha\frac{\dot{\tilde{k}}}{\tilde{k}}=g+\alpha\frac{\dot{\tilde{k}}}{\tilde{k}}$$

We know g doesn't change and we just showed that $\dot{\tilde{k}}/\tilde{k}<0$. Therefore, output per worker will grow at a lower rate initially. This rate may be negative if $\left|\alpha\dot{\tilde{k}}/\tilde{k}\right|>g$, but there is not enough information in the problem to determine if that is the case. We can, however, determine that the growth rate of y will approach the original growth rate in a convex way based on the graph shown earlier. The two cases are shown below (not necessarily to scale).



Therefore, the change permanently reduces the level of output per worker.

2. *An increase in the labor force*. Shocks to an economy, such as wars, famines, or the unification of two economies, often generate large one-time flows of workers across borders. What are the short-run and long-run effects on an economy of a one-time permanent increase in the stock of labor? Examine this question in the context of the Solow model with g = 0 and n > 0.



Steady State. The steady state growth rates are

$$\frac{\dot{y}}{y}=\frac{\dot{k}}{k}=0 \quad \text{and} \quad \frac{\dot{Y}}{Y}=\frac{\dot{K}}{K}=\frac{\dot{L}}{L}=n$$

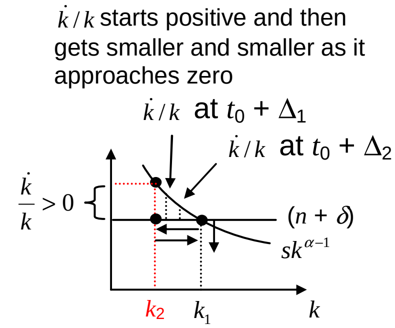
therefore, the one time change in the population (L) doesn't affect these rates.

Transition. Focus on two equations that are valid all the time (not just in steady state):

$$y=k^{\alpha} \quad \text{and} \quad \dot{k}=sy+(n+\delta)k$$

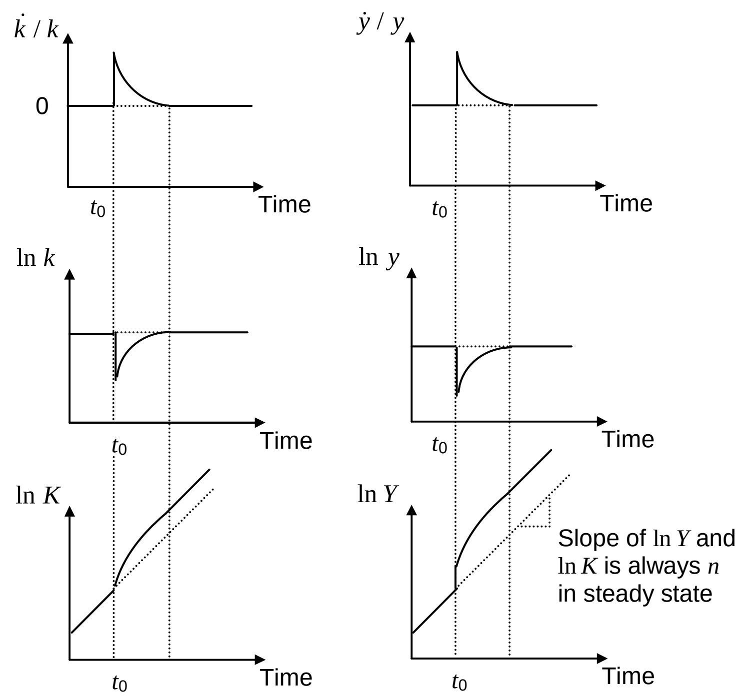
The second equation can be rewritten by substituting $y=k^{\alpha}$ and dividing both sides by k:

$$\frac{\dot{k}}{k} = \frac{s}{k^{1-\alpha}} - (n+\delta)$$



This helps us describe the transitional dynamics by plotting $sk^{\alpha-1}$ and $(n+\delta)$. The difference between the curves gives $\dot{k}/k$, the growth rate of capital per worker; in this case, the sudden increase in population causes an immediate drop in capital per worker (k). As the transition graph indicates, the lower value of k results in a positive growth rate in k. (Intuitively, what's happening is an increase in capital in order to return to the original level of capital per worker.)
To determine the effect on output per worker (y), do the ln-differentiate trick to the intensive form production function:

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$$



We just showed that $\dot{k}/k > 0$, so we also have $\dot{y}/y > 0$. According to the transition graph shown above, these growth rates will drop in a convex way until they return to zero.
To relate these growth rates to total capital (K) we look at the definition of k:

$$K = kL$$

We know L doesn't directly affect K, so this equation explains why k dropped initially. Do the ln-differentiate trick to get:

$$\frac{\dot{K}}{K} = \frac{\dot{k}}{k} + \frac{\dot{L}}{L} = \frac{\dot{k}}{k} + n$$

Since n is constant, we can ignore that and realize the change in the growth rate of capital is the same as the change in the growth rate of capital per worker.
To relate these growth rates to total output (Y) we look at the full production function:

$$Y = K^{\alpha} L^{1-\alpha}$$

Here we do have an initial effect form the change in L (i.e., Y increases at t_0).
Do the ln-differentiate trick to get:

$$\frac{\dot{Y}}{Y} = \alpha \frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} = \alpha \frac{\dot{K}}{K} + (1-\alpha)n$$

Again, n is constant so the change in the growth rate of output will mirror the change in the growth rate of capital.

3. *An income tax*. Suppose the U.S. Congress decides to levy an income tax on both wage income and capital income. Instead of receiving $wL + rK = Y$, consumers receive $(1 - \tau)wL + (1 - \tau)rK = (1 - \tau)Y$. Trace the consequences of this tax for output per worker in the short and long runs, starting from steady state.

$(1 - \tau)$ cancels so there's no difference in the production function. The per capita capital accumulation function does change:

$$\dot{k} = sy(1-\tau) - (n+\delta)k$$

Substitute $y = k^{\alpha}$ and divide both sides by k:

$$\frac{\dot{k}}{k} = \frac{s(1-\tau)}{k^{\alpha-1}} - (n+\delta)$$

Steady State.

$$\frac{\dot{k}}{k} = \frac{s(1-\tau)}{k^{\alpha-1}} - (n+\delta) = 0 \quad \Rightarrow \quad k^* = \left(\frac{s(1-\tau)}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

So the income tax effectively lowers the investment rate (same as problem 1).

4. *Manna falls faster*. Suppose that there is a permanent increase in the rate of technological progress, so that g rises to g'. sketch a graph of the growth rate of output per worker over time. Be sure to pay close attention to the transition dynamics.

Steady State. Increase in g causes stead state capital per worker (k) and output per worker (y) to increase to g'.

Transition. Focus on two equations that are valid all the time (not just in steady state):

$$\tilde{y} = \tilde{k}^{\alpha} \quad \text{and} \quad \dot{\tilde{k}} = s\tilde{y} - (n+g+\delta)\tilde{k}$$

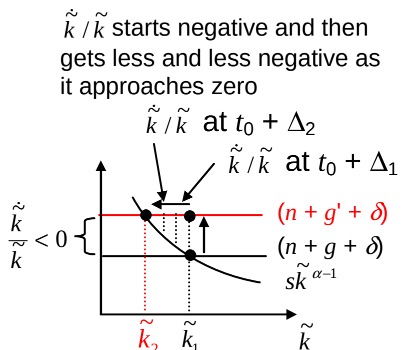
Divide the equation on the left by $\tilde{k}$ to see how g relates to the growth rate of $\tilde{k}$:

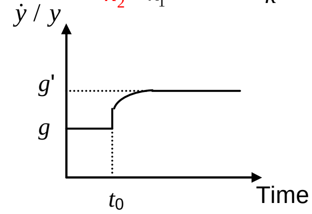
$$\frac{\dot{\tilde{k}}}{\tilde{k}} = s\tilde{k}^{\alpha-1} - (n+g+\delta)$$

Therefore, $\dot{\tilde{k}}/\tilde{k} = -\Delta g$ (as shown in the first diagram)

To figure out what happens to y, we use $\tilde{y} = y/A$ and the ln-differentiate trick we did before:

$$\frac{\dot{y}}{y} = \frac{\dot{A}}{A} + \alpha\frac{\dot{\tilde{k}}}{\tilde{k}} = g + \alpha\frac{\dot{\tilde{k}}}{\tilde{k}}$$





We see that when g increases to g', the growth rate of output per worker (y) increases, but the increase is less than Δg because of the drop in the growth rate of capital per effective worker which equals $-\Delta g$ (its effect doesn't totally counter Δg because $\alpha < 1$). Eventually the growth rate of y will increase to g' as shown in the second diagram.

1. Consider the following version of a Romer economy. There are only two sectors in the economy: The final goods sector and the research sector. Final output is produced according to the following production function:

$$Y = \ln H_Y \int_0^{A(t)} x(i)di$$

where H_Y is the amount of labor in the final goods sector and "$\ln$" denotes the natural logarithm; $A(t)$ is the measure of designs produced by time t, and x is the quantity of design used.

The production of designs uses labor and it is governed by the following equation:

$$\frac{\dot{A}(t)}{A(t)} = \delta H_A$$

Where δ is a productivity parameter, and H_A is the amount of labor engaged in research. The full employment condition of labor is given by

$$H = H_A + H_Y$$

where H is the fixed supply of labor in the economy.

Assume that the government buys every design produced in the research sector by paying a price P_A, which should be considered as a fixed parameter in the analysis. In addition, assume that only one unit of each design is needed for final output production, and therefore set $x = 1$. Perfect competition prevails in all markets and the government expenditure on designs is financed through grants from an international organization. There is no international trade in the model.

a. Solve for the balanced growth equilibrium assuming $x = 1$, and treating P_A as a parameter. Calculate the long-run equilibrium values of output growth, and the long-run allocation of labor in production H_Y and research H_A.
b. What are the effects of an increase in δ, H, and P_A on the balanced growth rate of output and on the growth rate of the wage of labor w?

a. Growth Rate

$$x = 1 \;\Rightarrow\; Y = \ln H_Y \int_0^{A(t)} x(i)di = A(t)\ln H_Y$$

Do the $\ln$-differentiate trick to get the growth rate of Y:

$$\ln Y = \ln A(t) + \ln(\ln H_Y)$$

$g_Y = \frac{\dot{Y}}{Y} = \frac{\dot{A}}{A} = g_A$ ... growth rate of economy is same as growth rate of $A(t)$

(H_Y drops out because it's constant at steady-state; if it wasn't the full employment or labor condition would imply that H_A go to zero [or H] because H is fixed.)

Labor Market

Workers move between research and manufacturing sectors, so in steady-state wages in these markets are equal.

Assume output Y is numeraire so $P_Y = 1$

Manufacturing: final good monopoly: $\max_{H_Y} \boldsymbol{\pi}_Y = A(t)\ln H_Y - w_Y H_Y - A(t)P_X$

FOC: $\frac{\partial \boldsymbol{\pi}_Y}{\partial H_Y} = \frac{A(t)}{H_Y} - w_Y = 0 \quad \Rightarrow \quad w_Y = \frac{A(t)}{H_Y}$

Another Way... marginal product of labor (MPL) $\frac{\partial Y}{\partial H_Y} = \frac{A(t)}{H_Y}$

$w_Y =$ value of MPL: $w_Y = P_Y \frac{\partial Y}{\partial H_Y} = (1)\frac{A(t)}{H_Y}$

Research: same as manufacturing, $w_A =$ value of MPL

Value of an innovation is P_A

Productivity of all researchers comes from $\frac{\dot{A}(t)}{A(t)} = \delta H_A \quad \Rightarrow \quad \dot{A}(t) = \delta A(t) H_A$

Differentiate with respect to H_A to get productivity per researcher: $\delta A(t)$

$\therefore w_A = \delta A(t) P_A$

Steady-State Conditions

(1) Growth of output... $g_Y = g_A = \frac{\dot{A}(t)}{A(t)} = \delta H_A$

(2) Labor market wage equilibrium... $w = \frac{A(t)}{H_Y} = \delta A(t) P_A$

(3) Labor market capacity... $H = H_A + H_Y$

Solve (2) for H_Y: $H_Y = \frac{A(t)}{\delta A(t) P_A} = \frac{1}{\delta P_A}$

Sub H_Y into this (3) and solve for H_A: $H_A = H - \frac{1}{\delta P_A}$

Sub H_A into (1): $g_Y = \delta\left(H - \frac{1}{\delta P_A}\right) = \delta H - \frac{1}{P_A}$

$$\boxed{\begin{aligned} g_Y &= \delta H - \frac{1}{P_A} \\ H_Y &= \frac{1}{\delta P_A} \\ H_A &= H - \frac{1}{\delta P_A} \end{aligned}}$$

b. Growth rate of output is given by $g_Y = \delta H - \frac{1}{P_A}$

Growth rate of wages comes from $w = \delta A(t) P_A$

Do the $\ln$-differentiate trick:

$\ln w = \ln \delta + \ln A(t) + \ln P_A$

$g_w = \frac{\dot{w}}{w} = \frac{\dot{A}}{A} = g_A = g_Y$ (because δ and P_A are constant in steady-state)

$\therefore$ g_Y and g_w will respond the same way to changes in δ, H, and P_A

Use $g_Y = \delta H - \frac{1}{P_A}$ ... results follow directly from this equation:

$\delta \uparrow \Rightarrow g_Y \uparrow$ (and $g_w \uparrow$)
$H \uparrow \Rightarrow g_Y \uparrow$ (and $g_w \uparrow$)
$P_A \uparrow \Rightarrow g_Y \uparrow$ (and $g_w \uparrow$)

2. The representative consumer problem in the typical growth model (i.e., Romer (1990)) can be stated as maximizing the discounted intertemporal utility function

$$\int_0^\infty \frac{C^{1-\sigma}}{1-\sigma} e^{-\rho t} dt$$

with respect to consumption expenditure $C(t)$, σ is the constant elasticity of substitution, and $\rho > 0$ is the constant subjective discount rate; The above expression is maximized subject to the intertemporal budget constraint

$$\dot{Z}(t) = r(t)Z(t) + w(t) - C(t)$$

where $Z(t)$ is the value of assets at time t, $r(t)$ is the instantaneous market interest rate, and $w(t)$ is the wage income at time t. Show that the solution to the intertemporal consumer maximization problem satisfies the following differential equation

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho}{\sigma}$$

State Variable: $Z(t)$; Control Variable: $C(t)$

<u>Present Value Hamiltonian</u> (evaluates objective at time 0)

Hamiltonian: $H = \frac{C^{1-\sigma}}{1-\sigma} e^{-\rho t} + \lambda(rZ + w - C)$

Properties of Optimal Solution: (come ECO 7406 Notes; KS p.127)

(1) $\frac{\partial H}{\partial C} = C^{-\sigma} e^{-\rho t} - \lambda = 0$ Optimality Condition

(2) $-\frac{\partial H}{\partial Z} = \lambda'(t) = -r\lambda$ Multiplier Equation

(3) $\frac{\partial H}{\partial \lambda} = Z'(t) = rZ + w - C$ State Equation

(4) $Z(0) = Z_0$ Boundary Condition

(5) $Z(\infty) = ?$ Boundary Condition

(6) $H_{CC} = -\sigma C^{-\sigma-1} e^{-\rho t} \leq 0$ for max Second Order Condition

Solve (2): $\lambda'(t) = -r\lambda \;\Rightarrow\; \lambda(t) = e^{-rt}$

Solve (1) for $C(t)$: $C = \left(\frac{e^{-\rho t}}{\lambda}\right)^{\frac{1}{\sigma}} = \left(\frac{e^{-\rho t}}{e^{-rt}}\right)^{\frac{1}{\sigma}} = \left(e^{-\rho t + rt}\right)^{\frac{1}{\sigma}}$

Do the ln-differentiate trick:

$$\ln C = \frac{1}{\sigma}\left(-\rho t - \ln\frac{1}{\lambda}\right) = \frac{1}{\sigma}\left(-\rho t + rt\right)$$

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left(-\rho + r\right) = \frac{r-\rho}{\sigma}$$

<u>Current Value Hamiltonian</u> (evaluates objective at time t)

Hamiltonian: $H = \frac{C^{1-\sigma}}{1-\sigma} + \lambda(rZ + w - C)$ (leaves off the $e^{-\rho t}$; no discounting here)

Properties of Optimal Solution: (given by Prof. Dinopoulos in class)

[1] $\frac{\partial H}{\partial C} = C^{-\sigma} - \lambda = 0$

[2] $\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial H}{\partial Z} = \rho\lambda(t) - r\lambda(t)$ (discount rate enters here)

Solve [2] for growth rate: $\frac{\dot{\lambda}(t)}{\lambda(t)} = \rho - r$

Solve [1] for C: $C = \lambda^{-\frac{1}{\sigma}}$

Do the ln-differentiate trick:

$$\ln C = -\frac{1}{\sigma}\ln\lambda$$

$$\frac{\dot{C}}{C} = -\frac{1}{\sigma}\frac{\dot{\lambda}}{\lambda} = -\frac{1}{\sigma}(\rho - r) = \frac{r-\rho}{\sigma}$$

3. Consider the following simple version of the Schumpeterian growth model described in Dinopoulos (1996) "Schumpeterian Growth Theory: An Overview." The utility of the representative consumer is given by

$$U(t) = \int_0^\infty e^{-3t} \ln(z(t))dt$$

where the sub-utility $z(t)$ is defined as $z(x_0, x_1, x_2, \ldots) = \sum_{q=0}^{\infty} (1.5)^q x_q$. There is only one industry in the economy, and the arrival of innovations is governed by a Poisson process with intensity L, where L is the amount of R&D researchers employed by firms engaged in sequential R&D races. The economy has $N = 27$ workers that can be allocated between the two activities, manufacturing and R&D. One worker can produce one unit of research services and/or one unit of final consumption good.

Use the wage of labor as the numeraire and calculate the stead-state equilibrium values of the following variables: the market interest rate $r(t)$; the equilibrium number of R&D workers L; consumption expenditure E; the amount of workers in manufacturing; the long-long run growth rate of the economy, g; the flow of profits at each instant in time $\boldsymbol{\pi}$; and the stock market valuation of monopoly profits $V(t)$.

Don't need to do derivation of general equilibrium conditions if we know them; they're included here for easy reference (and studying!)

Demand for $x_q = \begin{cases} \dfrac{E}{P_q} & \text{if } P_q \le \alpha w \\ 0 & \text{if } P_q > \alpha w \end{cases}$ (we substitute w (marginal cost) for P_{q-1})

To max profits firm will charge as much as possible while still driving competitor (maker of P_{q-1}) out of the market $\therefore P_q = \alpha w$

<u>Monopoly Profits</u>

$$\boldsymbol{\pi} = (P_q - w)x_q = (w\alpha - w)\frac{E}{w\alpha}$$

<u>Firm Doing R&D</u>

Pr(firm j making discovery) = $\mu_j dt = (L^\gamma)\dfrac{L_j}{L}dt = L_j L^{\gamma-1}dt$

Pr(discovery made) = $L^\gamma dt$

$V(t)$ is discounted earnings of winner of R&D race; to get expected discounted earnings, we have to multiply by probability of winning: $V(t)L_j L^{\gamma-1}dt$

Cost for R&D = $wL_j dt$

Expected Discounted Profits = $V(t)L_j L^{\gamma-1}dt - wL_j dt$

If we assume free entry into R&D Race, expected profits will be zero $\therefore$

$V(t)L_j L^{\gamma-1}dt - wL_j dt \Rightarrow V(t) = wL^{1-\gamma}$ (stock market value of the firm)

Stock Market Efficiency

Risk free return is $r(t)$; return on risk free bond in time dt is $r(t)dt$

Return for stock of firm that has monopoly on current good is sum of dividends plus expected capital games/losses (gains times Pr(firm survives) minus value of firm times Pr(firm disappears)):

$$\frac{\pi}{V(t)}dt + \frac{dV}{V(t)}(1 - L^{\gamma}dt) - \frac{V(t)-0}{V(t)}L^{\gamma}dt$$

Use this trick $dV = \frac{\partial V}{\partial t}dt = \dot{V}dt$

Efficiency says: $\frac{\pi}{V(t)}dt + \frac{\dot{V}dt}{V(t)}(1 - L^{\gamma}dt) - L^{\gamma}dt = r(t)dt$

dt's cancel: $\frac{\pi}{V(t)} + \frac{\dot{V}}{V(t)}(1 - L^{\gamma}dt) = r(t) + L^{\gamma}$

$\lim_{dt\to 0}$: $\frac{\pi}{V(t)} + \frac{\dot{V}}{V(t)} = r(t) + L^{\gamma}$

Solve for $V(t)$: $V(t) = \dfrac{\pi}{r(t) + L^{\gamma} - \dfrac{\dot{V}}{V(t)}}$

Consumer Maximization

$\max_E U = \int_0^{\infty} e^{-\rho t}\ln[z(\bullet)]dt$ s.t. $\dot{A}(t) = rA + w - E$

$z(x_0, x_1, x_2, \ldots) = \sum_{q=0}^{\infty}\alpha^q x_q$, but p_q is set so that none of the previous goods is purchased so we can use $z(\bullet) = \alpha^q x_q$; also since consumers only purchase good q, we know $x_q = \frac{E}{P_q} = \frac{E}{w\alpha} \Rightarrow z(\bullet) = \alpha^q \frac{E}{w\alpha} = \frac{\alpha^{q-1}E}{w}$, so now the objective is $\max_E U = \int_0^{\infty} e^{-\rho t}\ln\left[\frac{\alpha^{q-1}E}{w}\right]dt$

State Variable: $A(t)$; Control Variable: $E(t)$

Current Value Hamiltonian: $\ln\left[\frac{\alpha^{q-1}E}{w}\right] + \lambda(rA + w - E)$

[1] $\frac{\partial H}{\partial E} = \frac{1}{E} - \lambda = 0$

[2] $\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial H}{\partial A} = \rho\lambda(t) - r\lambda(t)$

Solve [2] for growth rate: $\frac{\dot{\lambda}(t)}{\lambda(t)} = \rho - r$

Solve [1] for E: $E = \lambda^{-1}$

Do the $\ln$-differentiate trick:

$\ln E = -\ln\lambda$

$\frac{\dot{E}}{E} = -\frac{\dot{\lambda}}{\lambda} = -(\rho - r) = r - \rho$

Equilibrium Conditions - General

(1) $\pi = (w\alpha - w)\frac{E}{w\alpha}$ (monopoly profits)

(2) $V(t) = wL^{1-\gamma}$ (free entry to R&E or zero profit condition)

(3) $V(t) = \frac{\pi}{r(t) + L^{\gamma} - \frac{\dot{V}}{V(t)}}$ (stock market portfolio efficiency condition)

(4) $\frac{\dot{E}}{E} = r(t) - \rho$ (consumer utility maximization)

(5) $\bar{N} = \frac{E}{\alpha} + L$ (full employment condition)

Equilibrium Conditions - For This Problem... $\gamma = 1$, $w = 1$, $\alpha = 1.5$, $\rho = 3$, $\bar{N} = 27$

[1] $\pi = (w\alpha - w)\frac{E}{w\alpha} = \frac{\alpha - 1}{\alpha}E = \frac{0.5}{1.5}E = \frac{1}{3}E$

[2] $V(t) = wL^{1-\gamma} = 1$

[3] $V(t) = \frac{\pi}{r(t) + L - \frac{\dot{V}(t)}{V(t)}}$

[4] $\frac{\dot{E}}{E} = r(t) - 3$

[5] $27 = \frac{2}{3}E + L$

Steady-state $\Rightarrow$ $\frac{\dot{E}}{E} = 0$ $\Rightarrow$ from [4] we have $r(t) = 3$

Since [3] says $V(t) = 1$, that means $\dot{V}(t) = 0$ $\therefore$ from [3] $V(t) = \frac{\pi}{3 + L}$

Sub [1]: $V(t) = \frac{\frac{1}{3}E}{3 + L} = 1$ $\Rightarrow$ $E = 9 + 3L$

Sub into [5]: $27 = \frac{2}{3}(9+3L)+L \;\;\Rightarrow\;\; 27 = 6+2L+L \;\;\Rightarrow\;\; L = 21/3 = 7$

Solve for E: $E = 9+3L = 9+3(7) = 30$

Solve for π: $\boldsymbol{\pi} = \frac{1}{3}E = \frac{1}{3}30 = 10$

Solve for $V(t)$: $V(t) = \dfrac{\boldsymbol{\pi}}{3+L} = \dfrac{10}{3+7} = 1$

Growth Rate $= L\ln\alpha = 7\ln 1.5 = 2.84$

$$\boxed{\begin{array}{l} r(t)=3 \\ L=7 \\ E=30 \\ L_M = 27-L=20 \\ g=2.84 \\ \boldsymbol{\pi}=10 \\ V(t)=1 \end{array}}$$