Review

Public Economics -
- **Broad** - study of government in modern economy; Hamilton: "a little bit like general surgery;" every decade or so, a new organ gets taken over by a specialist
- **Narrower** - what can/should government do to improve static and dynamic performance of the economy
- **Macroeconomics** - studies tax and expenditure policies; this is NOT what we'll do in public economics
- **Our Focus** - (1) optimal commodity taxation (2nd best); (2) optimal income taxation (classic asymmetric information problem)
- **Divisions** - Richard Musgrave (1950s) split field by branches of government
  - **Allocation** - what gets produced and consumed? Direct government action (taxation; public production); government influences on private choices (regulation)
    - Note: this doesn't correspond to traditional breakdown of tax and expenditure
  - **Distribution** - government influences on who consumes how much (tax policy)
  - **Stabilization** - dealt with in macro

Private Enterprise Economies - our focus for this course; government doesn't directly enter production; focus on government setting rules and providing incentives (or disincentives); Europeans are more concerned with public enterprise

Fundamental Theorems -
- **1st FTWE** - a competitive equilibrium is Pareto optimal
- **2nd FTWE** - any Pareto optimal allocation can be sustained as a competitive equilibrium after redistribution of endowments
- Hamilton: "if the conditions of these theorems applied most of the time, we wouldn't be teaching this course"
- **Failures** - monopoly, externalities, public goods
  - **Solution** - government intervention may restore efficiency or improve it (**2nd Best** - can't always get back to Pareto optimal outcome, but can maximize efficiency given some restrictions)

Roadmap - this is what we'll cover in this course
- **Welfare Economics** - welfare functions; interpersonal comparisons; public goods (sum of MRS = MRT)
- **Excess Burden Analysis** - measuring amount of inefficiency; doing dead weight loss correctly (done differently for IO)
- **Commodity Tax Reform** - local improvements from inefficiency allocations
- **Optimal Commodity Taxation** - full blown general equilibrium 2nd best model; answer tells a lot about monopoly regulation and overhead allocation for public enterprise; "how to collect fixed amount of revenue at minimum efficiency cost"
- **Optimal Income Taxation** - classic adverse selection problem; redistribution
- **If Time** - integrate optimal taxation and public goods (changes Samuelson rule); optimal taxation with uncertainty

Welfare Economics

Consumer Theory - studied information revealed by individual choices to derive complete theory of demand; done with ordinal (not cardinal) preferences so it wasn't comparable across individuals

- **Loosen Assumptions** - 1880s to 1960s, economists removed assumptions and got same results
- **Revealed Preference** - worry about consistency in choice; don't need to worry about type or form of utility functions

Welfare Economics - want to talk about good and bad allocations; dealing with "social preferences"

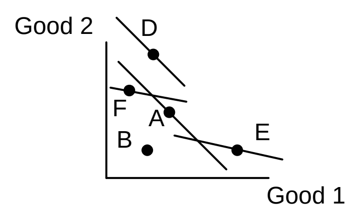
- **Allocation** - complete description of economy; "enough information for everyone concerned"
- **Limitations to FTWEs** -
  - **No Externalities** - individual preferences can only depend on <u>own</u> consumption (i.e., Len doesn't care what Josh consumes and vice versa); people are "selfish"; sounds negative, but also means there's no envy;
  - **No Public Goods** - no "shared consumption"

Individual Preferences - basic definitions

- **Weak Preference** - $\mathbf{x}\,\mathrm{R}_i\,\mathbf{y}$ ... person i ranks allocation $\mathbf{x}$ at least as good as $\mathbf{y}$
  - **Complete** - $\mathbf{x}\,\mathrm{R}_i\,\mathbf{y}$ or $\mathbf{y}\,\mathrm{R}_i\,\mathbf{x}$ or both
  - **Transitive** - $\mathbf{x}\,\mathrm{R}_i\,\mathbf{y}$ and $\mathbf{y}\,\mathrm{R}_i\,\mathbf{z} \;\Rightarrow\; \mathbf{x}\,\mathrm{R}_i\,\mathbf{z}$
- **Strict Preference** - $\mathbf{x}\,\mathrm{P}_i\,\mathbf{y} \equiv \mathbf{x}\,\mathrm{R}_i\,\mathbf{y}$ and Not $\mathbf{y}\,\mathrm{R}_i\,\mathbf{x}$
- **Indifference** - $\mathbf{x}\,\mathrm{I}_i\,\mathbf{y} \equiv \mathbf{x}\,\mathrm{R}_i\,\mathbf{y}$ and $\mathbf{y}\,\mathrm{R}_i\,\mathbf{x}$

Social Preferences - need to come up with Pareto criterion; we'll assume individual preferences count and are <u>complete</u> and <u>transitive</u>

- **Pareto Superior** - $\mathbf{x}\,\mathrm{P}\,\mathbf{y}$ if $\mathbf{x}\,\mathrm{R}_i\,\mathbf{y}\ \forall\ i$ and $\mathbf{x}\,\mathrm{P}_i\,\mathbf{y}$ for at least some i
  - Can also say $\mathbf{x}$ **Pareto dominates** $\mathbf{y}$; "unanimity rule" (if there is one person who does not at least weakly prefer $\mathbf{x}$ to $\mathbf{y}$, then $\mathbf{x}$ isn't Pareto superior to $\mathbf{y}$)
- **Pareto Optimal** - allocation $\mathbf{x}$ is Pareto optimal (**efficient**) if there does not exist an allocation $\mathbf{z}$ such that $\mathbf{z}\,\mathrm{P}\,\mathbf{x}$
  - **Problem** - PO criterion isn't very useful because there's no way to choose among PO allocations
  - **Revealed Preference** - look at data on aggregate consumption of two goods and price ratio when the choice is made; look at graph



    - A $\succ$ B   A has more of both goods so this is obvious (budget line when B is chosen is irrelevant)
    - D $\succ$ A   A is available when D is chosen and is not on the budget line
    - A $\succeq$ E or E $\succeq$ A   Neither is available when the other is chosen so we can't compare them
    - A & F   Inconsistent; can't draw tangent indifference curves through F and A without them intersecting

  (from ECO 7115 notes)

  - **Weak Axiom of Revealed Preference (WARP)** - proposed by Samuelson who argued this is equivalent to preferences satisfying standard properties; can't be done (need transitivity): If $\mathbf{x}$ is chosen under prices $\mathbf{P}'$ and $\mathbf{P}'\cdot\mathbf{x} \geq \mathbf{P}'\cdot\mathbf{y}$ (i.e., $\mathbf{y}$ is in the budget set) and if $\mathbf{y}$ is chosen under prices $\mathbf{P}''$, then $\mathbf{P}''\cdot\mathbf{y} < \mathbf{P}''\cdot\mathbf{x}$

Strong Axiom of Revealed Preference (SARP) - proposed by Houthakker; assumed transitivity holds which allows us to find all properties of preferences

Aggregate Consumption - there are too many possibilities; WARP doesn't work with redistribution

Utilitarianism - $W = u_1 + u_2$; assumes identical preferences and fixed incomes

New Welfare Economics - developed in 1930s; no cardinal utility or interpersonal comparisons, but still have Pareto criterion; trying to compare allocations to determine which is best

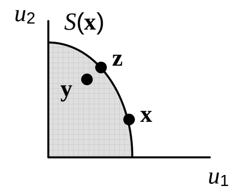
Notation - $\mathbf{x}$ and $\mathbf{y}$ are allocations that define "social states" (decisions we might make; e.g., build bridge type x or build bridge type y)

$S(\mathbf{x})$ is the set of allocaitons accessible from allocation $\mathbf{x}$ by redistribution

$S(\mathbf{y})$ is a different set of allocaitons accessible from allocaiton $\mathbf{y}$

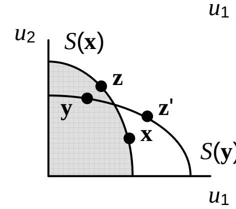
Note: this does <u>NOT</u> imply that $\mathbf{x}$ or $\mathbf{y}$ are Pareto optimal

Pareto Optimal - $\nexists\ \mathbf{y} \in S(\mathbf{x})$ such that $\mathbf{y}\ \mathrm{P}\ \mathbf{x}$; $\mathbf{x}$ is on the frontier of $S(\mathbf{x})$



Kaldor Superior - $\mathbf{x}\ \mathrm{K}\ \mathbf{y}$ means $\exists\ \mathbf{z} \in S(\mathbf{x})$ such that $\mathbf{z}\ \mathrm{P}\ \mathbf{y}$ (i.e., there is an allocation available by redistribution from $\mathbf{x}$ that Pareto dominates allocation $\mathbf{y}$)

Makes Sense? $\mathbf{x}\ \mathrm{K}\ \mathbf{y}$ seems reasonablle since moving from $\mathbf{y}$ to $\mathbf{x}$ has the potential to make everyone better off (by redistributing to get to allocaiton $\mathbf{z}$)

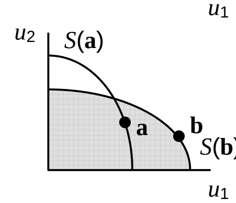


Problems - (1) compensation technically doesn't have to take place

(2) can have $\mathbf{x}\ \mathrm{K}\ \mathbf{y}$ and $\mathbf{y}\ \mathrm{K}\ \mathbf{x}$

Hamilton: this should have your "bullshit meter quivering"

Scitovsky's "Patch Job" - $\mathbf{b}\ \mathrm{S}\ \mathbf{a}$ means $\mathbf{b}\ \mathrm{K}\ \mathbf{a}$ and not $\mathbf{a}\ \mathrm{K}\ \mathbf{b}$



Problems - (1) relation can change based on endowment (start point); in third graph, $\mathbf{y}$ changed to $\mathbf{b}$ ($\mathbf{x}$ and $\mathbf{a}$ are same)

(2) still have cmpensation problem

Theorem - if $S(\mathbf{x}) = S(\mathbf{y})$ then $\mathbf{x}\ \mathrm{S}\ \mathbf{y}$ *iff* $\mathbf{x}$ is Pareto optimal and $\mathbf{y}$ is not

<u>Proof</u>: (a) $\mathbf{x}\ \mathrm{S}\ \mathbf{y} \Rightarrow \mathbf{x}$ is Pareto optimal and $\mathbf{y}$ is not

$\mathbf{x}\ \mathrm{S}\ \mathbf{y}$ means $\mathbf{x}\ \mathrm{K}\ \mathbf{y}$ and not $\mathbf{y}\ \mathrm{K}\ \mathbf{x}$

$\mathbf{x}\ \mathrm{K}\ \mathbf{y}$ means $\exists\ \mathbf{z} \in S(\mathbf{x}) = S(\mathbf{y})$ such that $\mathbf{z}\ \mathrm{P}\ \mathbf{y}$

$\therefore\ \mathbf{y}$ is not Pareto optimal

Suppose $\mathbf{x}$ is not Pareto optimal

That means $\exists\ \mathbf{w} \in S(\mathbf{x}) = S(\mathbf{y})$ such that $\mathbf{w}\ \mathrm{P}\ \mathbf{x}$ which means $\mathbf{y}\ \mathrm{K}\ \mathbf{x}$, but that contradicts $\mathbf{x}\ \mathrm{S}\ \mathbf{y}$ so $\mathbf{x}$ is Pareto optimal

(b) $\mathbf{x}$ is Pareto optimal and $\mathbf{y}$ is not $\Rightarrow\ \mathbf{x}\ \mathrm{S}\ \mathbf{y}$

$\mathbf{x}$ is Pareto optimal means $\nexists\ \mathbf{z} \in S(\mathbf{x}) = S(\mathbf{y})$ such that $\mathbf{z}\ \mathrm{P}\ \mathbf{x}$

That means not $\mathbf{y}\ \mathrm{K}\ \mathbf{x}$

$\mathbf{y}$ is not Pareto optimal means $\exists\ \mathbf{w} \in S(\mathbf{x}) = S(\mathbf{y})$ such that $\mathbf{w}\ \mathrm{P}\ \mathbf{y}$

That means $\mathbf{x}\ \mathrm{K}\ \mathbf{y}$

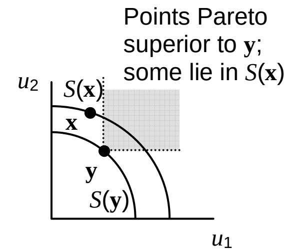
Since $\mathbf{x}\ \mathrm{K}\ \mathbf{y}$ and not $\mathbf{y}\ \mathrm{K}\ \mathbf{x}$, we have $\mathbf{x}\ \mathrm{S}\ \mathbf{y}$

<u>Problem?</u> - this theorem seems bogus; $S(\mathbf{x}) = S(\mathbf{y})$ and $\mathbf{x}\ \mathrm{S}\ \mathbf{y}$ cannot be possible; both $\mathbf{x}$ and $\mathbf{y}$ would have to be on the frontier in order for $S(\mathbf{x}) = S(\mathbf{y})$, but that would make both $\mathbf{x}$ and $\mathbf{y}$ Pareto optimal... after talking to Prof Hamilton, he said defining the feasible set ($S(\mathbf{x})$) doesn't mean $\mathbf{x}$ is on the frontier of the set

Theorem - if $S(\mathbf{x}) = S(\mathbf{y})$ then $\mathbf{x}\,\mathrm{K}\,\mathbf{y} \Rightarrow \mathbf{y}$ is not Pareto optimal

Problem - Kaldor and Scitovsky were well intentioned attempts to push things further (i.e., make more comparisons not available with just Pareto optimal criterion), but they don't work

Samuelson Superior - $\mathbf{x}\,\mathrm{N}\,\mathbf{y}$ means $\forall\ \mathbf{z} \in S(\mathbf{y})$, $\mathbf{x}\,\mathrm{K}\,\mathbf{z}$ (i.e., $S(\mathbf{y}) \subset S(\mathbf{x})$; $S(\mathbf{y})$ is completely contained in $S(\mathbf{x})$; any allocaiton that's possible from endowment $\mathbf{y}$ is possible from endowment $\mathbf{x}$)



Problems - (1) if $S(\mathbf{x}) = S(\mathbf{y})$, you can't have Samuelson superiority

(2) Still have compensation problem... but if the compensation is actually paid, Samuelson superiority guarantees that $S(\mathbf{x})$ has a better allocation than everything available with endowment $\mathbf{y}$ (i.e., can beat everything on the frontier of $S(\mathbf{y})$)

Revealed Preference - want to use aggregate data (revealed preference) to draw conclusions about social welfare; some forms work, but in general we can't do this

Community Indifference Curves - proposed by Scitovsky; curves satisfy three conditions (example based on two goods: x and y; and two individuals: 1 and 2)

Consume All Goods - total amount of each good available is consumed between all the individuals: $x = x^1 + x^2$ and $y = y^1 + y^2$

Constant Utility - each individual's utility level is held constant (although they could be different from each other): $u^1(x^1, y^1) = m^1$ and $u^2(x^2, y^2) = m^2$

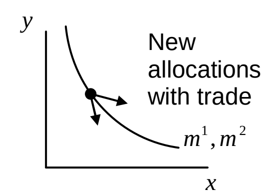
Same MRS - the marginal rate of substitution between the goods is the same for all individuals: $\mathrm{MRS}^1 = \mathrm{MRS}^2$

(From ECO 7120 notes)

$$\mathrm{MRS}^1 = \frac{\partial u^1 / \partial x^1}{\partial u^1 / \partial y^1}$$

MRS is the amount of good x a person has to give up when gaining a specified amount of good y in order to keep his utility constant

Community Indifference Curve (CIC) - level curves defined by m^1 and m^2 (and plotted in commodity space)



Minimum Total Requirements Curve - Samuelson's name for the same thing, but he defined it a little differently:

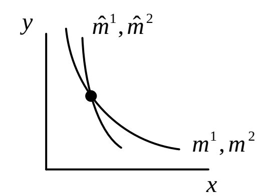
$$\max_{x^1, x^2, y^1, y^2} u^2(x^2, y^2)$$

s.t. $u^1(x^1, y^1) = m^1$, $x = x^1 + x^2$, $y = y^1 + y^2$

(This will result in $\mathrm{MRS}^1 = \mathrm{MRS}^2$ and gets to Pareto frontier of m^1, m^2 utility)

Income Distribution - changes in income distribution changes the MRS at all points so the CIC is only useful for a given income distribution... this means we can't use CID for revealed preference

Trade - problem with CIC; we can't tell if a new allocation achieved by trade is an improvement because of income distribution problem

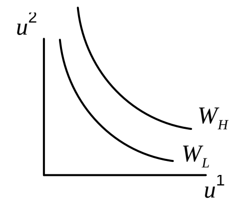


Social Welfare Function

Samuelson - said the "new welfare economics" had gone too far

Family Analogy - think of maximizing family welfare; think of family where all consumption is individual consumption; allocation of income isn't independent of prices (in effect, income distribution cushions price shocks that affect one member more than others to equalize their marginal utility of consumption)

Bergson-Samuelson Social Welfare Function (SWF) - $W(u^1(\mathbf{x}^1), u^2(\mathbf{x}^2), u^3(\mathbf{x}^3))$



Notes: (1) $\mathbf{x}^i$ is commodity vector consumed by individual i

(2) utilities depend only on own consumption

Paretian - $\frac{\partial W}{\partial u^i} > 0$ (if anybody's utility increases, social welfare increases)

Individualistic - $\frac{\partial W}{\partial x_j^i} = \frac{\partial W}{\partial u^i}\frac{\partial u^i}{\partial x_j^i}$ (change in SWF based on individual i's consumption of good j is only based on how i's utility changes and how that changes SWF... doesn't impact other individuals)

Graph - SWF defines indifference curves plotted in utility space

Problems -

Violates "new welfare economics" goals:

- Interpersonal comparisons added back in
- Added cardinal utility to "scale" individual utilities

Excludes some goals (when we max SWF s.t. resource constraints)

- Individualism excludes merit goods/paternalism (e.g., parents encourage kids to do "what's good for them"; or discourage "what's bad")
- Rules out specific egalitarianism - doesn't consider whether individuals get certain minimum amount (e.g., food, shelter, healthcare)
- Thrown out minimalist views of government - maximizing SWF requires redistribution

Stiglitz - "New new welfare economics"; economists job is to determine what's consistent with welfare maximization; policy makers determine the welfare function... or given welfare functions, economists determine whether policy is consistent with welfare maximization

Benefits - sounds bad, but we do get something from SWF... unlike Scitovsky's CIC, SWF lets us make trade offs between individuals

SWF Maximization Problem -

$$\max_{\mathbf{x}^1,\mathbf{x}^2} W(u^1(\mathbf{x}^1), u^2(\mathbf{x}^2))$$

s.t. $x_1^1 + x_1^2 = \bar{x}_1$ (λ_1) and $x_2^1 + x_2^2 = \bar{x}_2$ (λ_1) (resource constraints)

First Order Conditions -

(1) $\frac{\partial W}{\partial u^1}\frac{\partial u^1}{\partial x_1^1} - \lambda_1 = 0$

(2) $\frac{\partial W}{\partial u^2}\frac{\partial u^2}{\partial x_1^2} - \lambda_1 = 0$

(3) $\frac{\partial W}{\partial u^1}\frac{\partial u^1}{\partial x_2^1} - \lambda_2 = 0$

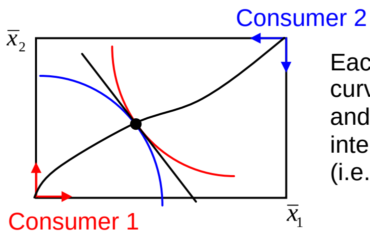
(4) $\frac{\partial W}{\partial u^2}\frac{\partial u^2}{\partial x_2^2} - \lambda_2 = 0$

First term in each of these is MU of individual's consumption times marginal impact (of individual's utility) on welfare

Take (1)/(3) and (2)/(4): $$\frac{\frac{\partial W}{\partial u^1}\frac{\partial u^1}{\partial x_1^1}}{\frac{\partial W}{\partial u^1}\frac{\partial u^1}{\partial x_2^1}} = \frac{\lambda_1}{\lambda_1} = \frac{\frac{\partial W}{\partial u^2}\frac{\partial u^2}{\partial x_1^2}}{\frac{\partial W}{\partial u^2}\frac{\partial u^2}{\partial x_2^2}}$$

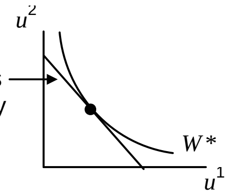
This simplifies to $\frac{\partial u^1/\partial x_1^1}{\partial u^1/\partial x_2^1} = \frac{\partial u^2/\partial x_1^2}{\partial u^2/\partial x_2^2}$ which means $\mathrm{MRS}^1 = \mathrm{MRS}^2$

Result - the solution to the unconstrained social welfare maximization problem is Pareto efficient (because $\mathrm{MRS}^1 = \mathrm{MRS}^2$), but we're also picking a particular point on the contract curve; we're choosing among Pareto efficient allocations to maximize SWF

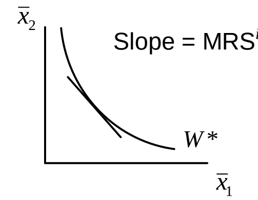


Each point on contract curve is allocation with u^1 and u^2 determined by intersecting indifferences (i.e., $\text{MRS}^1 = \text{MRS}^2$)

Utility possibilities frontier defined by contract curve



Maximized Value - $W*(\bar{x}_1, \bar{x}_2)$; since the only parameters in the SWF model are the resource constraints (given specific individual utility functions), the maximized value is defined by these resource constraints

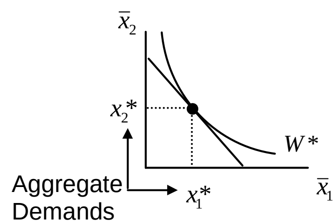


Level Curves - can draw level curves of $W*$ in consumption space similar to CIC, but doesn't take m^1 and m^2 as given; instead $W*$ is maximizing SWF for the given resource levels

Result - there's a unique curve through each point in resource space (unlike CIC)

Properties of $W*(\bar{x}_1, \bar{x}_2)$ - Samuelson 1956 and Gorman 1959; assuming:

(i) convex individual preferences
(ii) concave social welfare function (in utilities)
(iii) concave utility functions
(iv) lump sum transfers across individuals (i.e., pass income between individuals)



(1) Aggregate demands are functions of market prices and total income only (different than Scitovsky's CIC)
(2) Aggregate demands satisfy revealed preference properties
(3) Social indifference curves (contours of $W*$) satisfy all properties of individual indifference curves
(first two properties are most important)

Alternative SWF Maximization Problem - long way or "burry utility functions in SWF"

$$\max_{\mathbf{x}^1, \mathbf{x}^2} W(u^1(\mathbf{x}^1), u^2(\mathbf{x}^2)) \qquad \max_{\mathbf{x}^1, \mathbf{x}^2} \widetilde{W}(\mathbf{x}^1, \mathbf{x}^2)$$

s.t. $\mathbf{P} \cdot \mathbf{x}^1 + \mathbf{P} \cdot \mathbf{x}^2 = I^1 + I^2 \qquad \mathbf{P} \cdot (\mathbf{x}^1 + \mathbf{x}^2) = I^1 + I^2$

Significance - looks like regular individual utility maximization problem, except there are 2n goods rather than n goods (have to keep track of who consumes what)

Convexity of SWF Indiff Curves - proof of property (3) above

Pick two allocations A and B with same social welfare:

$$W(u^1(\mathbf{x}_A^1), u^2(\mathbf{x}_A^2)) = W(u^1(\mathbf{x}_B^1), u^2(\mathbf{x}_B^2))$$

Define allocation C that's half way between A and B: $\mathbf{x}_C^1 = \tfrac{1}{2}\left(\mathbf{x}_A^1 + \mathbf{x}_B^1\right)$ and $\mathbf{x}_C^2 = \tfrac{1}{2}\left(\mathbf{x}_A^2 + \mathbf{x}_B^2\right)$

Concavity of utility tells us: $u_C^1 \geq \tfrac{1}{2}\left(u_A^1 + u_B^1\right)$ and $u_C^2 \geq \tfrac{1}{2}\left(u_A^2 + u_B^2\right)$

Concavity of SWF:

$$W(u^1(\mathbf{x}_C^1), u^2(\mathbf{x}_C^2)) \geq W\left(\tfrac{1}{2}(u_A^1 + u_B^1), \tfrac{1}{2}(u_A^2 + u_B^2)\right) \geq \tfrac{1}{2}\left[W(u^1(\mathbf{x}_A^1), u^2(\mathbf{x}_A^2)) + W(u^1(\mathbf{x}_B^1), u^2(\mathbf{x}_B^2))\right]$$

Indirect Utility Functions - instead of $W(u^1(\mathbf{x}^1), u^2(\mathbf{x}^2))$, we can use indirect utility functions (as long as consumers face the same prices): $W(V^1(\mathbf{P}, I^1), V^2(\mathbf{P}, I^2))$

Result - max social welfare at point where social marginal utilities of income are the same between all individuals... that means ideal income distribution is dependent on market prices

Summary - can go from aggregate consumption data to welfare statements, but have to assume we're redistributing income as necessary to maximize social welfare (Bator article)

Public Goods

Non-Rival - part that matters to us since we're worried about efficiency
Non-Excludable - relates to paying for the good so it's not related to efficiency
Example -

$H =$ number of individuals

$x^i =$ private good consumed by individual i; using single private good, but could easily be a vector of private goods (single good just makes it easier)

$y =$ level of public good (everyone consumes the same amount by definition)

$u^i(x^i, y) =$ individual i's utility is based only on his consumption of the private good and the total amount of the public good

Pareto Problem - $\max\limits_{x^1 \ldots x^H, y} u(x^1, y)$

s.t. $u^j(x^j, y) \geq \bar{u}^j \quad , \; j = 2, \ldots, H \qquad (\lambda_j)$

$F\left(\sum_{i=1}^{H} x^i, y\right) \leq 0 \qquad (\mu)$ note that $x = \sum_{i=1}^{H} x^i$ (total private good)

First Order Conditions -

(1) $\frac{\partial u^1}{\partial y} + \sum_{j=2}^{H} \lambda_j \frac{\partial u^j}{\partial y} - \mu \frac{\partial F}{\partial y} = 0 \;\Rightarrow\; \frac{\partial u^1}{\partial y} + \sum_{j=2}^{H} \lambda_j \frac{\partial u^j}{\partial y} = \mu \frac{\partial F}{\partial y}$

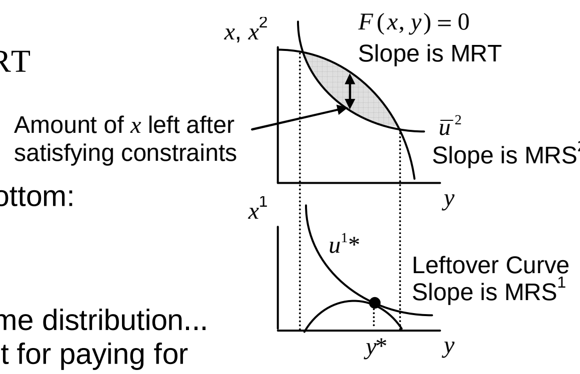
(2) $\frac{\partial u^1}{\partial x^1} - \mu \frac{\partial F}{\partial x} = 0$

(3) $\lambda_j \frac{\partial u^j}{\partial x^j} - \mu \frac{\partial F}{\partial x} = 0$

$$\frac{\partial u^1}{\partial x^1} = \lambda_j \frac{\partial u^j}{\partial x^j} = \mu \frac{\partial F}{\partial x}$$

$$\frac{\frac{\partial u^1}{\partial y}}{\frac{\partial u^1}{\partial x^1}} + \sum_{j=2}^{H} \frac{\lambda_j \frac{\partial u^j}{\partial y}}{\lambda_j \frac{\partial u^j}{\partial x^j}} = \frac{\mu \frac{\partial F}{\partial y}}{\mu \frac{\partial F}{\partial x}}$$

Samuelson Rule - $\sum_{j=1}^{H} \frac{\partial u^j / \partial y}{\partial u^j / \partial x^j} = \frac{\partial F / \partial y}{\partial F / \partial x} \;\Rightarrow\; \sum_{j=1}^{H} \mathrm{MRS}^j = \mathrm{MRT}$



Graphically - assume only 2 individuals (argument works for more)

Modify Samuelson Rule to go from top graph to bottom:

$\mathrm{MRS}^1 = \mathrm{MRT} - \mathrm{MRS}^2$

y^* and MRS^j depend on $\bar{u}^2$

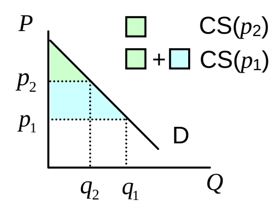
Result - level of public good (for efficiency) depends on income distribution... we'll have to modify the Samuelson rule when we account for paying for the public good

Measuring Efficiency Losses

Big Themes - of public economics:
- (1) lack of instruments - policies are limited because tools (like lump sum tax) don't exist
- (2) lack of information - can't always measure what something is worth to individuals

Excess Burden - measurement of inefficiency
- **Why** - (1) compare different tax systems; (2) determine if public good expenditure is worthwhile given the tax inefficiency that results from paying for it
- Lots of measures, but only one is correct

Tax Distortion - also called a wedge because producer and consumer prices are different
Value of Price Change - we'll focus on the price change first without regard to the tax revenue generated



<u>Consumer Surplus</u> - area under the demand curve minus expenditure (i.e., above price)

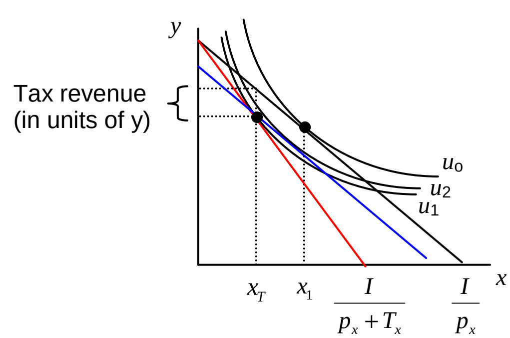
Move from p_1 to p_2 has ΔCS $= \int_{p_1}^{p_2} D(p)dp$ (the light blue area)

Problems -
- (1) **Utility Changes** as price changes; with tax some utility loss is inevitable, but want a clean measure (distinguish avoidable and unavoidable utility loss); this isn't as big a deal as problem 2
- (2) **Path Dependence** for multiple price changes; from picture, if we follow the blue path (change p_x first, then p_y) we get a different measure of ΔCS then if we take the red path (change p_y first, then p_x); problem is if we do a complete loop we can end up with ΔCS $\neq$ 0 (consumer better or worse off) even though there's really no net change

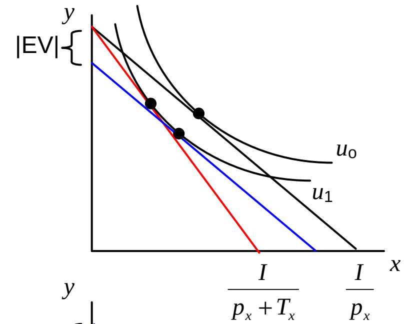
$$\begin{tikzcd} p_x^1, p_y^1 \arrow[r] \arrow[d] & p_x^2, p_y^1 \arrow[d] \\ p_x^1, p_y^2 \arrow[r] & p_x^2, p_y^2 \end{tikzcd}$$

Slutsky Equation - $\dfrac{\partial x^c}{\partial p_y} = \dfrac{\partial y^c}{\partial p_x}$ (partials of compensated demands wrt other price are equal); this is not true if there are income effects which is what causes the path dependence problem; if there are no income effects then path dependence isn't a problem

<u>Alternative</u> - EV and CV solve the path dependence problem; need some background first
- **Price Change** - if we raise the price of good x, the budget curve gets steeper (red line) and we move to a lower indifference curve (u_1)
- **Tax Revenue** - can be measured in units of y (since p_y is constant)
- **Lump Sum Tax** - can generate equal revenue with lump sum tax; lump sum holds relative prices constant so we just shift the original budget line through the new equilibrium (blue line); unless the indifference curve has a kink, utility will be higher with a lump sum tax (u_2); the difference between these utility levels is the inefficiency

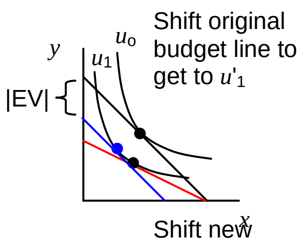


Problem - units of measurement (comparing utilities)
Solution - can get dollar measure by looking at equivalent income loss



Equivalent Variation - how much consumer is willing to pay to keep prices the same; how much income we can take away if we restore old prices and keep consumer at the same utility as with the price change

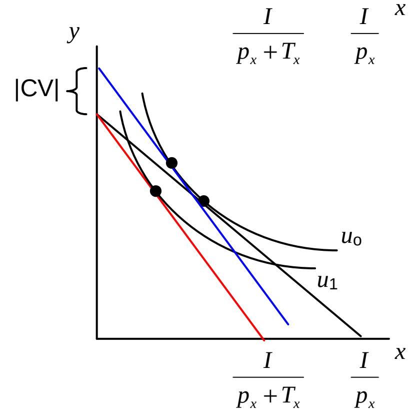
On Graph - move original budget line until tangent to new indifference curve (u_1)

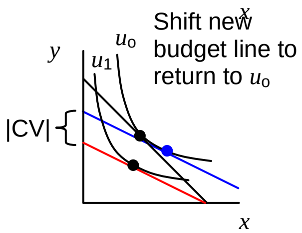


Formally - amount we need to give if change doesn't occur to make consumer as well of as if change had occurred (i.e., old prices with new utility)

EV ≡ $E(\mathbf{P}_o, u_1) - I = E(\mathbf{P}_o, u_1) - E(\mathbf{P}_1, u_1)$

$\mathbf{P}_1 > \mathbf{P}_0 \Rightarrow$ CV < 0





Compensating Variation - compensation needed to keep same utility with higher price; how much income we need to give the consume rot get him back to his original utility

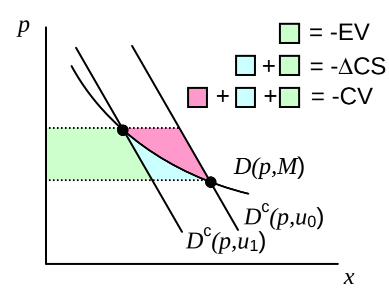
On Graph - move new budget line until tangent to original indifference curve (u_0)

Formally - amount of money we can take away after change leaving consumer as well off as before (i.e., income at new prices required to get old utility)

CV ≡ $I - E(\mathbf{P}_1, u_0) = E(\mathbf{P}_o, u_0) - E(\mathbf{P}_1, u_0)$... if $\mathbf{P}_1 > \mathbf{P}_0 \Rightarrow$ EV < 0

Which to Use - both measure the same thing (expenditure at old prices minus expenditure at new prices); only difference is which utility level (u_0 or u_1) is treated as the status quo; e.g., adding \$50 to \$50 is a 100% improvement, but losing \$50 from \$100 is only a 50% loss

Equivalence - CV of price change = -EV of reversing the change (Note: utilities are flipped in reverse: u_1 is "old" and u_0 is "new")

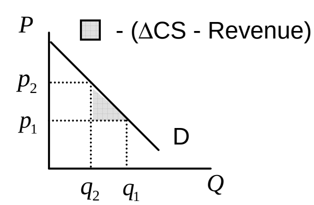


With Demand - assume x is a normal good (EV > CV)

Compensated Demand: if $u_0 > u_1$, then $D^c(p, u_0)$ is "above" $D^c(p, u_1)$

Regular Demand: function of price and money income; flatter than both compensated demands

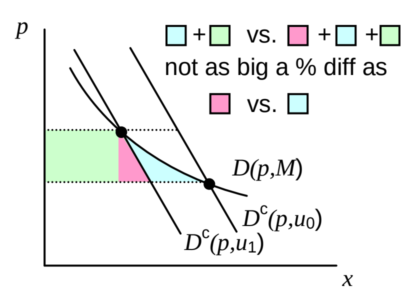
Price Change From Tax - for price changes in general, CV vs. EV isn't important (as long as you always use the same one); when dealing with price change from taxes however, you have to add back the revenue generated by the tax



Marshallian Deadweight Loss - uses ΔCS so it's problematic; still we know EV > ΔCS > CV so ΔCS looks nice

Willie (AER 1976) - "famous by obfuscation"; argued changes in CS (vs. EV or CV) tend to be small in practice (in % difference terms)

Problem - Hausman (AER 1981); all three have large area in common; it's like saying there's not much % error when weighing an elephant and a rabbit, but it's a big deal when you're worried about the weight of the rabbit



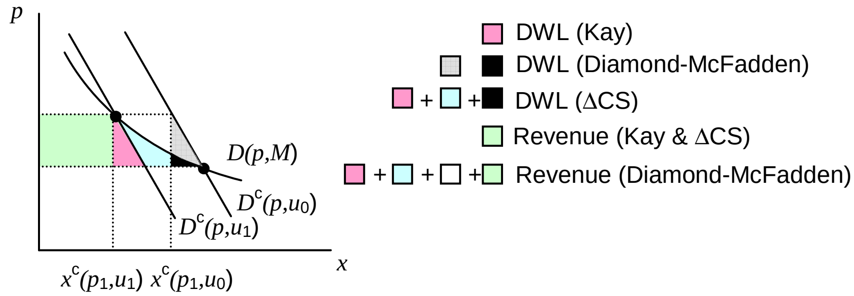
DWL Not Bounded - Hausman said when we account for revenue, dead weight loss using ΔCS is not bounded by EV and CV

Diamond-McFadden - (covered in Kay article); proposed measuring effect of tax by using $|CV| - T(\mathbf{P}_1, \mathbf{P}_0, u_0)$ where $\mathbf{P}_1$ = consumer prices; $\mathbf{P}_0$ = producer prices; u_0 = pre-tax utility

Tax Revenue Problem - $T(\mathbf{P}_1, \mathbf{P}_0, u_0)$ is supposed to measure of tax revenue, but it actually measures revenue if we paid compensation to make up for the change in prices caused by the tax; this is <u>infeasible</u> because compensation > revenue

With Actual Revenue - in graph below Diamond-McFadden DWL measured with actual tax revenue would be pink + blue + white + black + gray; this is clearly wrong in the graph, but some people don't catch it algebraically

Kay - updated Diamond-McFadden to use actual tax revenue: $|EV| - T(\mathbf{P}_1, \mathbf{P}_0, u_1)$



Kay's Notation -

$\mathbf{q} =$ after-tax price vector $\qquad$ $\mathbf{p} =$ pre-tax price vector

$\mathbf{x} =$ net purchases by consumer $\qquad$ $\mathbf{t} = \mathbf{q} - \mathbf{p} =$ tax

Move Axes - assume $E(\mathbf{q}, u_1) = E(\mathbf{p}, u_0) = 0$; this moves the intercept of the two budget lines to the origin; from the graphs, this is the point where the original budget line intercepts the new budget line (i.e., the intercept for the good whose price doesn't change; the non-taxed good)

Many Ways - to write excess burden (L):

$\bar{L} = -\mathrm{EV} - \mathbf{t}\cdot\mathbf{x}$ (Kay's definition)

$\bar{L} = E(\mathbf{q}, u_1) - E(\mathbf{p}, u_1) - \mathbf{t}\cdot\mathbf{x}$ (use $\mathrm{EV} = E(\mathbf{p}, u_1) - E(\mathbf{q}, u_1)$)

$\bar{L} = -E(\mathbf{p}, u_1) - \mathbf{t}\cdot\mathbf{x}$ (assuming $E(\mathbf{q}, u_1) = 0$)

$\bar{L} = \mathbf{p}\cdot\mathbf{x} - E(\mathbf{p}, u_1)$ (using $\mathbf{t} = \mathbf{q} - \mathbf{p}$ and $E(\mathbf{q}, u_1) = \mathbf{q}\cdot\mathbf{x}(\mathbf{q})$)

$\bar{L} = \mathbf{p}\cdot\{\mathbf{x}^c(\mathbf{q}, u_1) - \mathbf{x}^c(\mathbf{p}, u_1)\}$ (using $\mathbf{x} = \mathbf{x}^c(\mathbf{p}, u_1)$ $E(\mathbf{p}, u_1) = \mathbf{x}^c(\mathbf{p}, u_1)$)

Second Tax System - $\bar{L}' = E(\mathbf{q}', u_1') - E(\mathbf{p}, u_1') - \mathbf{t}'\cdot\mathbf{x}'$

Endogenize u_1 - use indirect utility function with $I = 0$:

$$\bar{L} = E(\mathbf{q}, V(\mathbf{q}, 0)) - E(\mathbf{p}, V(\mathbf{p}, 0)) - \mathbf{t}\cdot\mathbf{x}(\mathbf{q})$$

Properties of Excess Burden - correct measure is the Kay measure; properties:

(1) **Tax Improvement** - Shift in tax system which raises utility for same revenue must decrease excess burden (i.e., if new tax system makes people better off and government is collecting the same tax revenue, the new system is more efficient [less excess burden])

Change in Burden $= \bar{L}' - \bar{L} = E(\mathbf{q}', u_1') - E(\mathbf{p}, u_1') - \mathbf{t}'\cdot\mathbf{x}' - (E(\mathbf{q}, u_1) - E(\mathbf{p}, u_1) - \mathbf{t}\cdot\mathbf{x}) =$

$$\Delta\bar{L} = E(\mathbf{q}', u_1') - E(\mathbf{q}, u_1) + \mathbf{t}\cdot\mathbf{x} - \mathbf{t}'\cdot\mathbf{x}' + E(\mathbf{p}, u_1) - E(\mathbf{p}, u_1')$$

Since we moved the axes, $E(\mathbf{q}', u_1') = E(\mathbf{q}, u_1) = 0$

If we assume both tax systems raise the same revenue: $\mathbf{t}\cdot\mathbf{x} - \mathbf{t}'\cdot\mathbf{x}' = 0$

$\therefore\ \Delta\bar{L} = E(\mathbf{p},u_1) - E(\mathbf{p},u_1')$

$\Delta\bar{L} > 0 \Rightarrow E(\mathbf{p},u_1) - E(\mathbf{p},u_1') > 0$ (i.e., at same prices it takes more income to get to u_1 than to u_1')... that means $u_1 > u_1'$ so the tax system $\bar{L}$ is better than the system $\bar{L}'$

(2) **Optimal Tax** - minimizes excess burden

(3) **All Taxes Bad** - (a) all taxes have some inefficiency, (b) unless...

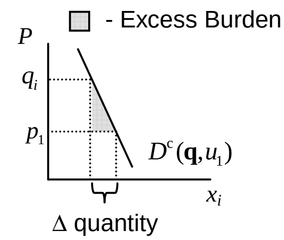
(3a) Since $u(\mathbf{x}(\mathbf{q})) = u_1$, $\bar{L} > 0$

(3b) $\bar{L} = 0$ only if compensated demand derivatives for all goods (wrt taxed commodities) equal zero; in this case commodity and lump sum taxes are equal

(4) $\left.\dfrac{\partial \bar{L}}{\partial q_i}\right|_{\mathbf{q}=\mathbf{p}} = 0$ at zero taxes (see (5) for case where taxes aren't zero);

Derivative of excess burden wrt any after tax price evaluated at before tax price is zero

English - excess burden is a second order property of a tax system; it's proportional to the square of taxes so as $q_i \to p_i$, $EB \to 0$ at a square of the rate



With Geometry - use area of a triangle (1/2 base times height):

$$EB \approx \frac{1}{2}(q_i - p_i)\left(\frac{\partial x_i^c}{\partial q_i}\cdot(q_i - p_i)\right) = \frac{1}{2}(q_i - p_i)^2\frac{\partial x_i^c}{\partial q_i},$$

(using a linear approximation of compensated demand)

With Taylor Series - look at change in excess burden going from $\mathbf{p}$ to $\mathbf{q}$

$$E(\mathbf{p},u_1) \approx E(\mathbf{q},u_1) + \mathbf{x}^c(\mathbf{q},u_1)(\mathbf{p}-\mathbf{q}) + \frac{1}{2}\mathbf{t}'\frac{\partial \mathbf{x}^c}{\partial \mathbf{q}}\mathbf{t} \quad \text{(Taylor series)}$$

$$\bar{L} = \mathbf{p}\cdot\mathbf{x} - E(\mathbf{p},u_1) = \mathbf{p}\cdot\mathbf{x} - \left(E(\mathbf{q},u_1) + \mathbf{x}^c(\mathbf{q},u_1)(\mathbf{p}-\mathbf{q}) + \frac{1}{2}\mathbf{t}'\frac{\partial \mathbf{x}^c}{\partial \mathbf{q}}\mathbf{t}\right)$$

Recall $E(\mathbf{q},u_1) = 0$: $\bar{L} = \mathbf{p}\cdot\mathbf{x} - (\mathbf{p}-\mathbf{q})\cdot\mathbf{x} - \dfrac{1}{2}\mathbf{t}'\dfrac{\partial \mathbf{x}^c}{\partial \mathbf{q}}\mathbf{t}$

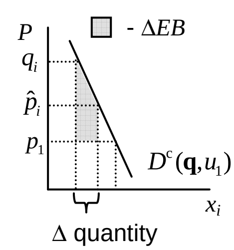
Note $\mathbf{q}\cdot\mathbf{x} = E(\mathbf{q},u_1) = 0$: $\bar{L} = -\dfrac{1}{2}\mathbf{t}'\dfrac{\partial \mathbf{x}^c}{\partial \mathbf{q}}\mathbf{t}$

This is a quadratic form so the sign of $\bar{L}$ depends on the sign of $\dfrac{\partial \mathbf{x}^c}{\partial \mathbf{q}}$

$\dfrac{\partial \mathbf{x}^c}{\partial q_i} \le 0\ \therefore\ \bar{L} \ge 0$ ... also note, $\bar{L}$ grows at the square of the tax ($\mathbf{t} = \mathbf{q} - \mathbf{p}$)

Consequence - better to tax many goods rather than just one; one big EB area will be bigger than many small ones

(5) **Harberger's Formula** - $\bar{L} = -\dfrac{1}{2}\sum_i\sum_j t_i t_j \dfrac{\partial x_j(\mathbf{q},u_1)}{\partial q_i}$



Initial Distortion - EB is a second order term (shown in (4))

Added Distortion - add tax to $\hat{p}_i$ that already has a distortion (e.g., monopoly or other tax) and we can have marginal EB > 0 and EB is no longer a second order term (goes to zero linearly as $q_i \to \hat{p}_i$)

Key to Kay - use compensated demands and after tax utility

Commodity Tax Reform

Narrow Problem - when lump-sum taxes are infeasible, what set of commodity taxes raises a target level of revenue and leaves consumers as well off as possible
Fixed Revenue - fixed for now; later we'll reverse the problem and consider how much revenue can be raise for a given level of inefficiency (which we'll use to determine if a public good should be funded... have to account for inefficiency that results from tax)
More Generally - study optimal commodity taxation has other benefits:
(1) Direct vs. Indirect Taxation
Direct Taxes - based on property, wealth, income, factor income
Benefits - (i) can tax different people at different rates; (ii) factors are inelastically supplied while indirect taxes tend to cause consumption distortions
Labor-Leisure - some argue labor supply isn't inelastic when factoring labor/leisure choice instead of just labor supply
Indirect Taxes - based on business transactions (e.g., sales tax, excise tax, tariffs)
History - federal government moved from indirect to direct (mainly through 16th Amendment which legalized income tax with redistribution)
Hamilton: "not a lot to be gained in distinction of direct vs. indirect"... just history
(2) Value-of-Service Pricing
Railroad Rates - Interstate Commerce Commission established in 1880s to regulate railroad rates; farmers pushed for regulation to lower shipping rates for farm output; R/R have high fixed cost which need to be allocated so rates exceed marginal cost; ICC decided to discriminate over commodities and keep rates low for commodities with highly elastic demand (i.e., not farm output or coal)
(3) General Theory of Second Best (Lipsey & Lancaster, 1957)
Said we shouldn't try to counter distortions if they can't be removed
(4) Regulated Utility Pricing - Boitteux
How do you want to set prices as a multiproduct monopolist; Boitteux worked for France's state-owned utility company looking at different types of customers
Formal Treatment - of optimal commodity tax
Ramsey, 1927
Samuelson, 1951 (unpublished)
Boitteux, 1956 (in French)
Diamond & Mirrlees, 1971 (explicit general equilibrium framework)... what we'll study
Stiglitz & Dasgupta, 1971
Baumol & Bradford, 1970 (specialized to utility pricing)

Tax Reform - Dixit (1975 & 1977); follows theory of 2nd best; what can we do to get local improvement... use duality, but requires lots of conditions so in real world it's hard to guarantee if change will be an improvement
Commodities - $n+1$ of them
Good Zero - numeraire; think of it as minus labor supply; $p_0 = 1,\ q_0 = 1,\ t_0 = 0$ (untaxed)... "normalization" (explained in Diamond-Mirrlees)
Commodity Vector - $(x_0, \mathbf{x}) = (x_0, x_1, x_2, \ldots, x_n)$
Consumer Prices - $\mathbf{q} = (q_1, \ldots, q_n)$
Producer Prices - $\mathbf{p} = (p_1, \ldots, p_n)$
Taxes - $\mathbf{t} = \mathbf{q} - \mathbf{p}$ (difference between what consumers pay and what producers receive

Production Cost - $C(\mathbf{x}) =$ numeraire cost of producing $\mathbf{x}$ (think of all output using only labor, x_0)

Marginal Cost - $\mathbf{C_x} = \left(\frac{\partial C}{\partial x_1}, \frac{\partial C}{\partial x_2}, \ldots \frac{\partial C}{\partial x_n} \right)$

$\mathbf{C_{xx}} = \left(\frac{\partial^2 C}{\partial x_i \partial x_j} \right)$, $n \times n$ matrix; assume positive semidefinite (allows fixed cost, but otherwise convex technology

Single Consumer - can think of it as many identical consumers

$E(1, \mathbf{q}, u) =$ minimum expenditure at prices $(1, \mathbf{q})$ to get utility u

Compensated Demands -

$x_0 = E_0(1, \mathbf{q}, u) = \frac{\partial E}{\partial q_0}$ (don't forget $\mathbf{q}$ = after-tax prices the consumer pays)

$\mathbf{x} = E_{\mathbf{q}}(1, \mathbf{q}, u) = \left(\frac{\partial E}{\partial q_i} \right)$

Assume $E_{\mathbf{qq}}$ is negative definite (rules out kinked demand curves)

Homogeneity of Degree Zero - $E_{i0} + \sum_{j=1}^{n} q_j E_{ij} = 0$, $i = 0, 1, \ldots n$

Note: $E_{i0} = \frac{\partial^2 E}{\partial q_i \partial q_0} = \frac{\partial x_i^c}{\partial q_0} = \frac{\partial x_0^c}{\partial q_i}$

Vector notation: $E_{00} + \mathbf{q}' \cdot E_{0\mathbf{q}} = 0$ and $E_{\mathbf{q}0}' + \mathbf{q}' \cdot E_{\mathbf{qq}} = \mathbf{0}$

Endowment - of numeraire is Z

Government - collects commodity taxes and a lump-sum tax to raise amount of G of numeraire good

Budget Constraint - $\mathbf{t}'\mathbf{x} + T = G$ (commodity tax revenue + lump-sum revenue $= G$)

Producer Profit - $P = \mathbf{p}'\mathbf{x} - C(\mathbf{x})$

Distributed as lump-sum to consumer

Consumer Budget - $Z - T + P = E(1, \mathbf{q}, u)$

Sub in gov't budget constraint: $Z - (G - \mathbf{t}'\mathbf{x}) + P = E(1, \mathbf{q}, u)$

Sub in producer profit: $Z - G + \mathbf{t}'\mathbf{x} + (\mathbf{p}'\mathbf{x} - C(\mathbf{x})) = E(1, \mathbf{q}, u)$

Solve for Z: $Z = G - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{x} + C(\mathbf{x}) + E(1, \mathbf{q}, u)$

Sub $\mathbf{t}'\mathbf{x} = (\mathbf{q} - \mathbf{p})'\mathbf{x}$ (cancels $-\mathbf{p}'\mathbf{x}$): $Z = G - \mathbf{q}'\mathbf{x} + C(\mathbf{x}) + E(1, \mathbf{q}, u)$

Sub $\mathbf{x} = E_{\mathbf{q}}(1, \mathbf{q}, u)$: $\boxed{Z = G + E(1, \mathbf{q}, u) - \mathbf{q}' E_{\mathbf{q}}(1, \mathbf{q}, u) + C(E_{\mathbf{q}}(1, \mathbf{q}, u))}$

Sub $E_0 = E(1, \mathbf{q}, u) - \mathbf{q}' E_{\mathbf{q}}(1, \mathbf{q}, u)$: $\boxed{Z = G + E_0 + C(E_{\mathbf{q}}(1, \mathbf{q}, u))}$

English - consumer endowment of labor (T) = government spending (G) + amount kept by consumer (leisure; E_0) + amount consumed in production ($C(E_{\mathbf{q}}(1, \mathbf{q}, u))$)

Totally differentiate (first box above): T and G are constant

$0 = E_{\mathbf{q}}' d\mathbf{q} - d\mathbf{q}' E_{\mathbf{q}} - \mathbf{q}' E_{\mathbf{qq}} d\mathbf{q} + \mathbf{C_x}' E_{\mathbf{qq}} d\mathbf{q} + (E_{0u} + \mathbf{C_x}' E_{\mathbf{q}u}) du$

Combine terms: $0 = E_{\mathbf{q}}' d\mathbf{q} - d\mathbf{q}' E_{\mathbf{q}} + (\mathbf{C_x} - \mathbf{q})' E_{\mathbf{qq}} d\mathbf{q} + (E_{0u} + \mathbf{C_x}' E_{\mathbf{q}u}) du$

Note $E_{\mathbf{q}}'d\mathbf{q} = d\mathbf{q}'E_{\mathbf{q}}$: $0 = (\mathbf{C_x} - \mathbf{q})'E_{\mathbf{qq}}d\mathbf{q} + (E_{0u} + \mathbf{C_x}'E_{\mathbf{q}u})du$

Solve for du: $du = \dfrac{(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q}}{E_{0u} + \mathbf{C_x}'E_{\mathbf{q}u}} = \dfrac{(1\times n)(n\times n)(n\times 1)}{1} =$ scalar

Know $E_{0u} + \mathbf{q}'E_{\mathbf{q}u} = \dfrac{\partial E}{\partial u} > 0$ (resource cost at consumer prices increases as utility increases) $\therefore$ assume $E_{0u} + \mathbf{C_x}'E_{\mathbf{q}u} > 0$ (i.e., resource cost at producer prices increases as utility increases)

$\therefore$ du has the same sign as $(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q}$

What Are We Doing? - looking at changing consumer prices (rather than tax; actually same as controlling tax with fixed producer prices) and seeing whether consumers are better off (du > 0) or worse off (du < 0); assume we adjust the lump-sum tax to maintain government revenue so consumer utility is not affected by changes government purchases

Better Off - want to find sufficient conditions to guarantee $(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q}$ > 0 (so we have du > 0... consumers are better off)

Reduce Tax (Price) - reducing tax on good i means $dq_i < 0$

***n* = 2 Case** - $(\mathbf{q} - \mathbf{C}_x)'E_{\mathbf{qq}}d\mathbf{q} = \begin{bmatrix} q_1 - c_1 \\ q_2 - c_2 \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} =$

$$\begin{bmatrix} q_1 - c_1 & q_2 - c_2 \end{bmatrix} \begin{bmatrix} E_{11}dq_1 + E_{12}dq_2 \\ E_{21}dq_1 + E_{22}dq_2 \end{bmatrix} =$$

$$(q_1 - c_1)E_{11}dq_1 + (q_1 - c_1)E_{12}dq_2 + (q_2 - c_2)E_{21}dq_1 + (q_2 - c_2)E_{22}dq_2$$

Tax One Good - suppose $q_2 = c_2$ (i.e., consumer price for good 2 equals marginal cost; that implies price taking on producer side and no taxes on good 2); this means $dq_2 = 0$ and $q_2 - c_2 = 0$ so the expression above simplifies to one term:

$(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q} = (q_1 - c_1)E_{11}dq_1$

$q_1 - c_1 > 0$ (since good 1 is taxed, the consumer price must be greater than MC)

$E_{11} < 0$ (because we assumed $E_{\mathbf{qq}}$ is negative definite)

That means if $dq_1 < 0 \Rightarrow du > 0$

English - if only one good is taxed, lowering the tax (i.e., price) on that good toward MC, then consumers are better off; makes sense and is the same result form the partial equilibrium set up (i.e., want price = MC)

Tax Both Goods - Now we have $q_1 - c_1 > 0$ and $q_2 - c_2 > 0$; assume we hold price of good 2 constant (i.e., $dq_2 = 0$; don't change the tax); now the expression above simplifies to two terms

$(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q} = (q_1 - c_1)E_{11}dq_1 + (q_2 - c_2)E_{21}dq_1$

If we lower the price of good 1 ($dq_1 < 0$), we know from previous case that the first term will be negative

The second case is ambiguous:

$q_2 - c_2 > 0$ and $dq_1 < 0$, but E_{21} can go either way

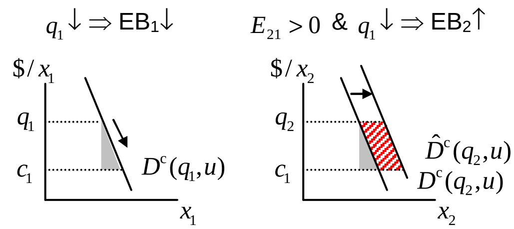
Another way to look at it is to factor dq_1:

$$(\mathbf{q} - \mathbf{C_x})' E_{\mathbf{qq}} d\mathbf{q} = \left[(q_1 - c_1)E_{11} + (q_2 - c_2)E_{21}\right] dq_1$$

We now the first term in brackets is negative and dq_1 is negative; if the second term is sufficiently positive, we could end up with the expression overall being negative (i.e., if $E_{21} >> 0$, $dq_1 < 0 \;\; \Rightarrow \;\; du < 0$)

$\therefore$ lowering the price of good one may or may not make consumers better off

Contradiction? - that sounds weird, but what's happening when we lower the price of good 1 there's a shift in the compensated demand for good 2; if E_{21} is positive, the shift is to the right (or up) which increases excess burden for good 2 while lowering q_1 decreases excess burden for good 1



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$(\mathbf{q} - \mathbf{C_x})' E_{\mathbf{qq}} d\mathbf{q} = (\mathbf{q} - \mathbf{C_x})' d\mathbf{x}^c$, which means the sign of du depends on how compensated demands change

Propositions - these are conditions to guarantee $du > 0$ (key is looking for patterns in the "distortion" (i.e., tax or price change: $\mathbf{q} - \mathbf{C_x}$) or in $E_{\mathbf{qq}}$

(1) **Proportional Change** - if consumer prices move toward marginal cost in proportion to existing distortions and the lump-sum tax changes to hold revenue constant then welfare rises

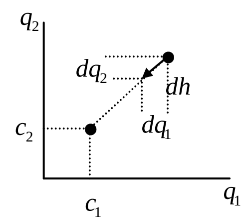
The proportional change means $d\mathbf{q} = -(\mathbf{q} - \mathbf{C_x})dh$, where dh is a scalar (measures the length of vector in graph; distance traveled by price change) so now the sign of du is the same as the sign of

$$(\mathbf{q} - \mathbf{C_x})' E_{\mathbf{qq}} d\mathbf{q} = -\underbrace{(\mathbf{q} - \mathbf{C_x})' E_{\mathbf{qq}} (\mathbf{q} - \mathbf{C_x})}_{\text{Symmetric quadratic form}} dh$$

$E_{\mathbf{qq}}$ is neg. def. so this product is < 0

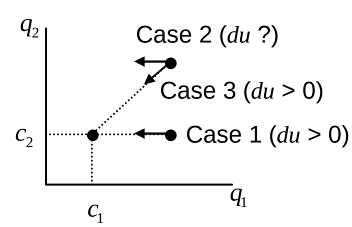
Proportional Change

Wherever we start, we move toward (c_1, c_2)



We have (-)(-)(+) so $du > 0$

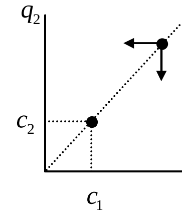
Revisit Examples - the first case on previous page (Tax One Good) satisfies this proportional change proposition (price change moves toward marginal cost); the second case (Tax Both Goods) doesn't because we're only lowering the tax on one good (i.e., not a proportional change toward marginal cost)



(2) **Corlett-Hague Theorem** - holding revenue constant through commodity taxes (i.e., $(\mathbf{q} - \mathbf{C_x})'\mathbf{x} = \overline{R}$; they didn't use lump-sum tax like Dixit did; that makes this more general and realistic than previous proposition), if prices are initially above MC by same factor for all goods (i.e., $q_i - c_i = \beta c_i$ [proportional to c_i]), welfare increases with <u>small</u> increases in prices of commodities complementary to the numeraire and small decreases in prices

Corlett-Hague Thm

Initially on ray thru origin and (c_1, c_2)



Move in either direction, but can't go far ("small changes") because it moves off proportional line and theorem no longer holds of commodities that are substitutes for numeraire; so sign of du is the same as the sign of $(\mathbf{q}-\mathbf{C_x})' E_{\mathbf{qq}} d\mathbf{q} = -\beta \mathbf{q}' E_{\mathbf{qq}} d\mathbf{q}$ (scalar)(1x*n*)(*n*x*n*)(*n*x1)

Apply homogeneity of degree zero assumption: $E_{00} + \mathbf{q}' \cdot E_{0\mathbf{q}} = 0$ and $E_{\mathbf{q}0}' + \mathbf{q}' \cdot E_{\mathbf{qq}} = \mathbf{0}$

$$-\beta \mathbf{q}' E_{\mathbf{qq}} d\mathbf{q} = \beta E_{\mathbf{q}0}' d\mathbf{q}$$

Sign of this term depends on sign of $E_{\mathbf{q}0}' d\mathbf{q}$ which is taken care of in the assumption:

$E_{i0} > 0$ (substitutes) $\Rightarrow dq_i < 0$

$E_{i0} < 0$ (complements) $\Rightarrow dq_i > 0$

$\therefore \beta E_{\mathbf{q}0}' d\mathbf{q} > 0$ so $du > 0$

Improvement - although this is restrictive in that we have to start with proportional distortions, we don't have to use a proportional tax (as long as there are substitutes and complements to the numeraire)

(3) **General** - given an arbitrary initial price vector; lowering q_j toward its marginal cost increases welfare if...

(a) good *j* is complementary to all goods with greater proportional distortions
(b) good *j* is substitute for all other goods (including numeraire)
(c) we adjust lump-sum taxes to hold revenue constant

<u>Proof</u>: recall sign of du will be same as sign of $(\mathbf{q}-\mathbf{C_x})' E_{\mathbf{qq}} d\mathbf{q}$

Label proportion of distortions with $\beta_j = \dfrac{q_j - c_j}{q_j}$

(similar to Corlett-Hague Thm except each good can have a different distortion)

Sort commodities by distortion so $\beta_1 \le \beta_2 \le \ldots \le \beta_j \le \ldots \le \beta_n$

We are only changing the price of good *j* so $dq_j < 0$ and $dq_k = 0$ for $k \ne j$

That means $(\mathbf{q}-\mathbf{C_x})' E_{\mathbf{qq}} d\mathbf{q}$ becomes $\sum_{i=1}^{n} (q_i - c_i) E_{ij} dq_j = \sum_{i=1}^{n} \beta_i q_i E_{ij} dq_j$

Use homogeneity of degree 0 again: $E_{j0} + \sum_{i=1}^{n} q_i E_{ij} = 0$

We can rewrite this as $\beta_j \left(E_{j0} + \sum_{i=1}^{n} q_i E_{ij} \right) dq_j = 0$

Now subtract it from $\sum_{i=1}^{n} \beta_i q_i E_{ij} dq_j$

$$\sum_{i=1}^{n} \beta_i q_i E_{ij} dq_j - \beta_j \left(E_{j0} + \sum_{i=1}^{n} q_i E_{ij} \right) dq_j =$$

$$\left[\sum_{i=1}^{n} (\beta_i - \beta_j) q_i E_{ij} - \beta_j E_{j0} \right] dq_j$$

To sign this, remember $dq_j < 0$; (a) and (b) take care of the rest

$E_{ij} > 0$ (substitutes) $\Rightarrow \beta_i - \beta_j \le 0$ (this include numeraire so $E_{j0} > 0$)

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Good with Largest Distortion - special case where $j = n$; lowering the price lessens excess burden if good n is a substitute for everything... that's a strong condition; if there are any complementary relationships, we can't guarantee lowering price would lower excess burden (it can, we just can't guarantee it)

Summary - common 2nd best result: reducing one distortion isn't obviously good (i.e., not clearly an improvement) when other distortions exist
Hamilton: "In the case of multiple distortions, tread carefully with your intuition."

If we know compensated demand derivatives ($E_{\mathbf{qq}}$) we can get clearer results

Commodity Tax Reform

Narrow Problem - when lump-sum taxes are infeasible, what set of commodity taxes raises a target level of revenue and leaves consumers as well off as possible

- **Fixed Revenue** - fixed for now; later we'll reverse the problem and consider how much revenue can be raise for a given level of inefficiency (which we'll use to determine if a public good should be funded... have to account for inefficiency that results from tax)

More Generally - study optimal commodity taxation has other benefits:

- (1) Direct vs. Indirect Taxation
  - **Direct Taxes** - based on property, wealth, income, factor income
    - **Benefits** - (i) can tax different people at different rates; (ii) factors are inelastically supplied while indirect taxes tend to cause consumption distortions
    - **Labor-Leisure** - some argue labor supply isn't inelastic when factoring labor/leisure choice instead of just labor supply
  - **Indirect Taxes** - based on business transactions (e.g., sales tax, excise tax, tariffs)
  - **History** - federal government moved from indirect to direct (mainly through 16th Amendment which legalized income tax with redistribution)
  - Hamilton: "not a lot to be gained in distinction of direct vs. indirect"... just history
- (2) Value-of-Service Pricing
  - **Railroad Rates** - Interstate Commerce Commission established in 1880s to regulate railroad rates; farmers pushed for regulation to lower shipping rates for farm output; R/R have high fixed cost which need to be allocated so rates exceed marginal cost; ICC decided to discriminate over commodities and keep rates low for commodities with highly elastic demand (i.e., not farm output or coal)
- (3) General Theory of Second Best (Lipsey & Lancaster, 1957)
  - Said we shouldn't try to counter distortions if they can't be removed
- (4) Regulated Utility Pricing - Boitteux
  - How do you want to set prices as a multiproduct monopolist; Boitteux worked for France's state-owned utility company looking at different types of customers

Formal Treatment - of optimal commodity tax

- Ramsey, 1927
- Samuelson, 1951 (unpublished)
- Boitteux, 1956 (in French)
- Diamond & Mirrlees, 1971 (explicit general equilibrium framework)... what we'll study
- Stiglitz & Dasgupta, 1971
- Baumol & Bradford, 1970 (specialized to utility pricing)

Tax Reform - Dixit (1975 & 1977); follows theory of 2nd best; what can we do to get local improvement... use duality, but requires lots of conditions so in real world it's hard to guarantee if change will be an improvement

- **Commodities** - $n+1$ of them
  - **Good Zero** - numeraire; think of it as minus labor supply; $p_0 = 1$, $q_0 = 1$, $t_0 = 0$ (untaxed)... "normalization" (explained in Diamond-Mirrlees)
  - **Commodity Vector** - $(x_0, \mathbf{x}) = (x_0, x_1, x_2, \ldots, x_n)$
- **Consumer Prices** - $\mathbf{q} = (q_1, \ldots, q_n)$
- **Producer Prices** - $\mathbf{p} = (p_1, \ldots, p_n)$
- **Taxes** - $\mathbf{t} = \mathbf{q} - \mathbf{p}$ (difference between what consumers pay and what producers receive

Production Cost - $C(\mathbf{x}) =$ numeraire cost of producing $\mathbf{x}$ (think of all output using only labor, x_0)

Marginal Cost - $\mathbf{C_x} = \left(\frac{\partial C}{\partial x_1}, \frac{\partial C}{\partial x_2}, \ldots \frac{\partial C}{\partial x_n} \right)$

$\mathbf{C_{xx}} = \left(\frac{\partial^2 C}{\partial x_i \partial x_j} \right)$, $n \times n$ matrix; assume positive semidefinite (allows fixed cost, but otherwise convex technology

Single Consumer - can think of it as many identical consumers

$E(1, \mathbf{q}, u) =$ minimum expenditure at prices $(1, \mathbf{q})$ to get utility u

Compensated Demands -

$x_0 = E_0(1, \mathbf{q}, u) = \frac{\partial E}{\partial q_0}$ (don't forget $\mathbf{q}$ = after-tax prices the consumer pays)

$\mathbf{x} = E_{\mathbf{q}}(1, \mathbf{q}, u) = \left(\frac{\partial E}{\partial q_i} \right)$

Assume $E_{\mathbf{qq}}$ is negative definite (rules out kinked demand curves)

Homogeneity of Degree Zero - $E_{i0} + \sum_{j=1}^{n} q_j E_{ij} = 0$, $i = 0, 1, \ldots n$

Note: $E_{i0} = \frac{\partial^2 E}{\partial q_i \partial q_0} = \frac{\partial x_i^c}{\partial q_0} = \frac{\partial x_0^c}{\partial q_i}$

Vector notation: $E_{00} + \mathbf{q}' \cdot E_{0\mathbf{q}} = 0$ and $E_{\mathbf{q}0}' + \mathbf{q}' \cdot E_{\mathbf{qq}} = \mathbf{0}$

Endowment - of numeraire is Z

Government - collects commodity taxes and a lump-sum tax to raise amount of G of numeraire good

Budget Constraint - $\mathbf{t}'\mathbf{x} + T = G$ (commodity tax revenue + lump-sum revenue $= G$)

Producer Profit - $P = \mathbf{p}'\mathbf{x} - C(\mathbf{x})$

Distributed as lump-sum to consumer

Consumer Budget - $Z - T + P = E(1, \mathbf{q}, u)$

Sub in gov't budget constraint: $Z - (G - \mathbf{t}'\mathbf{x}) + P = E(1, \mathbf{q}, u)$

Sub in producer profit: $Z - G + \mathbf{t}'\mathbf{x} + (\mathbf{p}'\mathbf{x} - C(\mathbf{x})) = E(1, \mathbf{q}, u)$

Solve for Z: $Z = G - \mathbf{t}'\mathbf{x} - \mathbf{p}'\mathbf{x} + C(\mathbf{x}) + E(1, \mathbf{q}, u)$

Sub $\mathbf{t}'\mathbf{x} = (\mathbf{q} - \mathbf{p})'\mathbf{x}$ (cancels $-\mathbf{p}'\mathbf{x}$): $Z = G - \mathbf{q}'\mathbf{x} + C(\mathbf{x}) + E(1, \mathbf{q}, u)$

Sub $\mathbf{x} = E_{\mathbf{q}}(1, \mathbf{q}, u)$: $\boxed{Z = G + E(1, \mathbf{q}, u) - \mathbf{q}' E_{\mathbf{q}}(1, \mathbf{q}, u) + C(E_{\mathbf{q}}(1, \mathbf{q}, u))}$

Sub $E_0 = E(1, \mathbf{q}, u) - \mathbf{q}' E_{\mathbf{q}}(1, \mathbf{q}, u)$: $\boxed{Z = G + E_0 + C(E_{\mathbf{q}}(1, \mathbf{q}, u))}$

English - consumer endowment of labor (T) = government spending (G) + amount kept by consumer (leisure; E_0) + amount consumed in production ($C(E_{\mathbf{q}}(1, \mathbf{q}, u))$)

Totally differentiate (first box above): T and G are constant

$$0 = E_{\mathbf{q}}' d\mathbf{q} - d\mathbf{q}' E_{\mathbf{q}} - \mathbf{q}' E_{\mathbf{qq}} d\mathbf{q} + \mathbf{C_x}' E_{\mathbf{qq}} d\mathbf{q} + (E_{0u} + \mathbf{C_x}' E_{\mathbf{q}u}) du$$

Combine terms: $0 = E_{\mathbf{q}}' d\mathbf{q} - d\mathbf{q}' E_{\mathbf{q}} + (\mathbf{C_x} - \mathbf{q})' E_{\mathbf{qq}} d\mathbf{q} + (E_{0u} + \mathbf{C_x}' E_{\mathbf{q}u}) du$

Note $E_{\mathbf{q}}'d\mathbf{q} = d\mathbf{q}'E_{\mathbf{q}}$: $0 = (\mathbf{C_x} - \mathbf{q})'E_{\mathbf{qq}}d\mathbf{q} + (E_{0u} + \mathbf{C_x}'E_{\mathbf{q}u})du$

Solve for du: $du = \dfrac{(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q}}{E_{0u} + \mathbf{C_x}'E_{\mathbf{q}u}} = \dfrac{(1\times n)(n\times n)(n\times 1)}{1} =$ scalar

Know $E_{0u} + \mathbf{q}'E_{\mathbf{q}u} = \dfrac{\partial E}{\partial u} > 0$ (resource cost at consumer prices increases as utility increases) $\therefore$ assume $E_{0u} + \mathbf{C_x}'E_{\mathbf{q}u} > 0$ (i.e., resource cost at producer prices increases as utility increases)

$\therefore$ du has the same sign as $(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q}$

What Are We Doing? - looking at changing consumer prices (rather than tax; actually same as controlling tax with fixed producer prices) and seeing whether consumers are better off (du > 0) or worse off (du < 0); assume we adjust the lump-sum tax to maintain government revenue so consumer utility is not affected by changes government purchases

Better Off - want to find sufficient conditions to guarantee $(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q}$ > 0 (so we have du > 0... consumers are better off)

Reduce Tax (Price) - reducing tax on good i means $dq_i < 0$

***n* = 2 Case** - $(\mathbf{q} - \mathbf{C}_x)'E_{\mathbf{qq}}d\mathbf{q} = \begin{bmatrix} q_1 - c_1 \\ q_2 - c_2 \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \end{bmatrix} =$

$$\begin{bmatrix} q_1 - c_1 & q_2 - c_2 \end{bmatrix} \begin{bmatrix} E_{11}dq_1 + E_{12}dq_2 \\ E_{21}dq_1 + E_{22}dq_2 \end{bmatrix} =$$

$$(q_1 - c_1)E_{11}dq_1 + (q_1 - c_1)E_{12}dq_2 + (q_2 - c_2)E_{21}dq_1 + (q_2 - c_2)E_{22}dq_2$$

Tax One Good - suppose $q_2 = c_2$ (i.e., consumer price for good 2 equals marginal cost; that implies price taking on producer side and no taxes on good 2); this means $dq_2 = 0$ and $q_2 - c_2 = 0$ so the expression above simplifies to one term:

$(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q} = (q_1 - c_1)E_{11}dq_1$

$q_1 - c_1 > 0$ (since good 1 is taxed, the consumer price must be greater than MC)

$E_{11} < 0$ (because we assumed $E_{\mathbf{qq}}$ is negative definite)

That means if $dq_1 < 0 \Rightarrow du > 0$

English - if only one good is taxed, lowering the tax (i.e., price) on that good toward MC, then consumers are better off; makes sense and is the same result form the partial equilibrium set up (i.e., want price = MC)

Tax Both Goods - Now we have $q_1 - c_1 > 0$ and $q_2 - c_2 > 0$; assume we hold price of good 2 constant (i.e., $dq_2 = 0$; don't change the tax); now the expression above simplifies to two terms

$(\mathbf{q} - \mathbf{C_x})'E_{\mathbf{qq}}d\mathbf{q} = (q_1 - c_1)E_{11}dq_1 + (q_2 - c_2)E_{21}dq_1$

If we lower the price of good 1 ($dq_1 < 0$), we know from previous case that the first term will be negative

The second case is ambiguous:

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Another way to look at it is to factor dq_1:

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We now the first term in brackets is negative and dq_1 is negative; if the second term is sufficiently positive, we could end up with the expression overall being negative (i.e., if $E_{21} >> 0$, $dq_1 < 0 \;\Rightarrow\; du < 0$)

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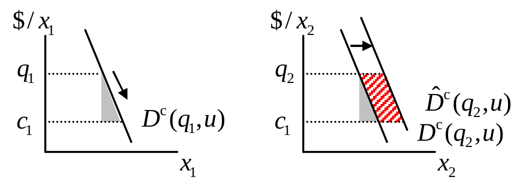
$q_1 \downarrow \Rightarrow$ EB$_1\downarrow$ $\qquad$ $E_{21} > 0$ & $q_1 \downarrow \Rightarrow$ EB$_2\uparrow$



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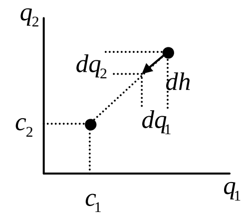

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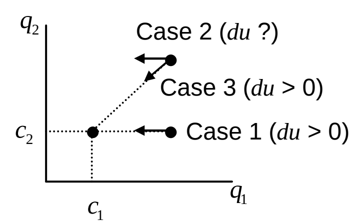

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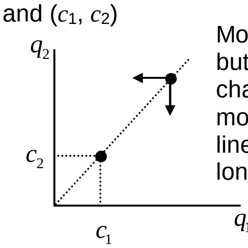

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$$\sum_{i=1}^{n} \beta_i q_i E_{ij} dq_j - \beta_j \left(E_{j0} + \sum_{i=1}^{n} q_i E_{ij} \right) dq_j =$$

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To sign this, remember $dq_j < 0$; (a) and (b) take care of the rest

$E_{ij} > 0$ (substitutes) $\Rightarrow \beta_i - \beta_j \le 0$ (this include numeraire so $E_{j0} > 0$)

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Hamilton: "In the case of multiple distortions, tread carefully with your intuition."

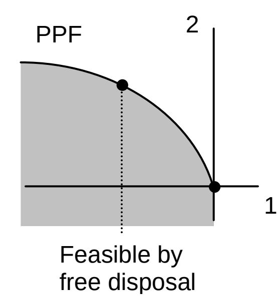
If we know compensated demand derivatives ($E_{\mathbf{qq}}$) we can get clearer results

Optimal Commodity Taxation

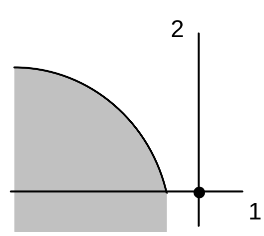
Previous section discussed changing commodity taxes to reduce excess burden (i.e., making consumers better off), but assumed there were lump-sum taxes to keep revenue constant
This section will assume we don't have lump-sum taxes; we'll solve a general equilibrium model that gives us "bonus" answers:
(1) How should government evaluate operations of its activities when lump-sum taxation is infeasible
(2) How should taxation that's used affect governments operating decisions
(3) How to levy taxes for 2nd best outcome (i.e., lump-sum taxes not available)
Goal - maximize social welfare subject to revenue and production constraints; decision variables include taxes and public production
Public Production - government can buy finished goods for public consumption or can produce goods itself; that means government buys output and sells output; since it's producing it is competing with private sector for factors of production
Areas - we're combining taxation, public production and welfare economics (and GE)

Consumer Budget Constraint - $\mathbf{q} \cdot \hat{\mathbf{x}} = \mathbf{q} \cdot \boldsymbol{\omega}$ (value of consumption = value of endowment);
problem is we don't see consumption, just net trades ($\mathbf{x}$): $\mathbf{x} + \boldsymbol{\omega} = \hat{\mathbf{x}}$
Sub this into the budget constraint: $\mathbf{q} \cdot (\mathbf{x} + \boldsymbol{\omega}) = \mathbf{q} \cdot \boldsymbol{\omega} \;\; \Rightarrow \;\; \mathbf{q} \cdot \mathbf{x} = 0$
(This allows us to suppress income when we go to multiple consumers)

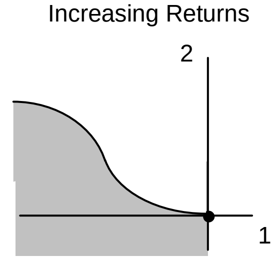
Want to end up with economy of many consumers with public and private production; want to compare different tax systems; want to determine types/amount of public consumption
Training Wheels - we'll start with single consumer, no public consumption (but no private production; all public production), and only use commodity taxes; 1 consumer and 2 goods (1 = labor; 2 = consumption good)
Weird Graphs -
Define origin as "no net trades" (i.e., consumer keeps his endowment)
Assume free disposal (any point below the PPF is feasible)
Consumer sells labor which is used for production
Problems - special cases don't have convex feasible regions, but are realistic so we won't assume them away
Fixed Cost - PPF doesn't cross origin, but origin is feasible
Increasing Returns - reason for natural monopolies



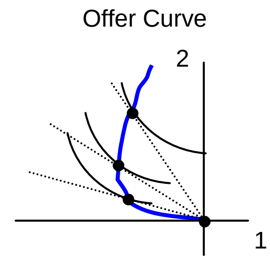
Fixed Cost



Increasing Returns



Other Assumptions -



- **No Lump-Sum Tax** - government can't tax endowments
- **Uniform Prices** - government can only trade with consumer in market
- **Price Consumption Curve** - all trades government makes with consumer lie on consumer's offer curve: tangencies of consumer budget line and indifference curves); all budget lines go through origin (endowment); moving along offer curve away from origin means consumer is better off

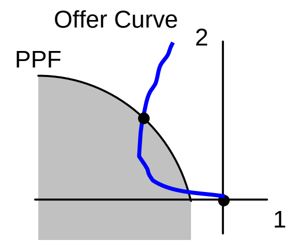
Two Constraints - government chooses point on offer curve lying in (on) PPF

More Than Two Goods

Consumer Demands - $\mathbf{x} = (x_1, x_2, \ldots, x_n)$

Where $\mathbf{x}(\mathbf{q},0)$ solves $\max_{\mathbf{x}} u(\mathbf{x})$ s.t. $\mathbf{q} \cdot \mathbf{x} = 0$ ($\mathbf{q}$ is consumer prices; 0 is money income)

PPF - defined by $G(\mathbf{z}) \leq 0$, where $\mathbf{z}$ is public production



Government Objective - $V(\mathbf{q})$ (technically it's $\tilde{V}(\mathbf{x}(\mathbf{q}))$; government chooses taxes so it can effectively change consumer prices

$$\max_{\mathbf{q}} V(\mathbf{q}) \text{ s.t. } G(\mathbf{x}(\mathbf{q})) \leq 0$$

Note: this ensures $\mathbf{x}(\mathbf{q}) = \mathbf{z}$ (i.e., demand = supply)

Individualistic - if government's concern is social welfare and SW is individualistic (based on consumer utility), then $V(\mathbf{q})$ increases as $u(\mathbf{x})$ increases; government's objective essentially becomes maximizing consumer's utility; solution is last intersection between offer curve and boundary of PPF ∴ we get production efficiency (trivial result for this economy)

Add Private Production

Private Sector - constant returns to scale
Public Sector - government runs any industry with increasing returns or fixed costs
Decreasing Returns - assume we have none

Individualistic - $\frac{\partial V}{\partial q_i} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial q_i} = -\alpha x_i$, where α is marginal utility of income to society

$$\therefore \frac{\partial V / \partial q_i}{\partial V / \partial q_j} = \frac{-\alpha x_i}{-\alpha x_j} = \frac{x_i}{x_j}$$

$\mathbf{p} = (p_1, p_2, \ldots, p_n) =$ producer prices

$\mathbf{y} = (y_1, y_2, \ldots, y_n) =$ quantities of private production (> 0 is output; < 0 is input)

$\mathbf{z} = (z_1, z_2, \ldots, z_n) =$ quantities of public production

Single Firm - if prices are given, constant returns allows us to consider the private sector as being a single firm

Private Production Efficient - we'll assume $y_1 = f(y_2, \ldots, y_n)$ (production is efficient)

Also assume f is differentiable and $y_i \neq 0 \ \forall \ i = 1, \ldots, n$ (i.e., private firm is either a net producer or consumer of each good)

Private Objective - $\max_{\mathbf{y}} \mathbf{p}\cdot\mathbf{y}$ s.t. $y_1 = f(y_2,\ldots,y_n)$ (λ)

Note: since $y_i > 0$ is output and $y_i < 0$ is input, $\mathbf{p}\cdot\mathbf{y}$ is profit

1st Order Conditions - $p_1 - \lambda = 0$ and $p_i - \lambda \dfrac{\partial f}{\partial y_i} = 0$

Notation: $f_i = \dfrac{\partial f}{\partial y_i}$ ... we can combine 1st order conditions: $p_i = p_1 f_i \ \forall \ i = 2,\ldots,n$

Note: $\mathbf{p}\cdot\mathbf{y} = 0$ otherwise problem would be unbounded $\therefore$ if we know $y_2,\ldots,y_n$ we know y_1 and $\mathbf{p}/p_1$ (ratio of prices relative to good 1)

Public Production Efficient - used to have $G(\mathbf{z}) \le 0$ for public (government) constraint, but now we'll use $z_1 = g(z_2,\ldots,z_n)$

Production Efficiency - we assumed private production is efficient and public production is efficient, but we didn't say anything about them being jointly efficient (i.e., $\mathrm{MRT}^{\mathrm{P}} = \mathrm{MRT}^{\mathrm{G}}$)

Consumer - used to have $\mathbf{x}(\mathbf{q}) = \mathbf{z}$, but now it becomes $\mathbf{x}(\mathbf{q}) = \mathbf{y} + \mathbf{z}$ (market clearing; demand = supply)

Walras' Law - since we have market clearing, if we know $n-1$ markets clear, we know all n markets clear because all agents must satisfy budget constraints

Reverse - we showed all n markets clear $\therefore$ if we know all but one agent satisfy their budget constraints, then they all do (so we can delete one budget constraint); in this case, we'll delete the government's budget constraint: $(\mathbf{q}-\mathbf{p})\cdot\mathbf{x} + \mathbf{p}\cdot\mathbf{z} = 0$

Normalizing - we get to do two of them because we have $\mathbf{x}(\mathbf{q})$ homogeneous of degree zero in $\mathbf{q}$ and $\mathbf{y}(\mathbf{p})$ homogeneous of degree zero in $\mathbf{p}$

(1) **Producer Price** - from the private production optimization we looked at earlier, we know only the price ratios ($\mathbf{p}/p_1$) matter so we'll set $p_1 = 1$

(2) **Consumer Price** - conventional choice is $q_1 = 1$

Tax Problem? - since taxes are $\mathbf{t} = \mathbf{q} - \mathbf{p}$, this looks like we just said good 1 is not taxed, but it's just a result of the normalization, not a model restriction... don't blink or you might miss the explanation...

$x_i > 0 \Rightarrow$ good purchased by the consumer

$x_i < 0 \Rightarrow$ good provided by the consumer; in this case $t_i > 0$ is actually a subsidy because $(q_i - p_i)\cdot x_i < 0$ (i.e., it costs the government money)

Proportional Tax - desirable because it keeps all the $\mathrm{MRS}_{ij} = \mathrm{MRT}$ (doesn't distort consumption), but if tax is proportional to producer prices for all goods, then it raises zero revenue

<u>Proof</u>: proportional tax means $\tau = \dfrac{\mathbf{q}-\mathbf{p}}{\mathbf{p}}$ or $\tau\mathbf{p} = \mathbf{q}-\mathbf{p}$ or $(1+\tau)\mathbf{p} = \mathbf{q}$

From consumer budget constraint: $\mathbf{q}\cdot\mathbf{x} = 0$

Proportional tax: $\mathbf{q} = (1+\tau)\mathbf{p}$

Substitute into budget constraint: $(1+\tau)\mathbf{p}\cdot\mathbf{x} = 0$

We can't have $\tau = -1$ so that means $\mathbf{p}\cdot\mathbf{x} = 0$

$\therefore$ $(\mathbf{q}-\mathbf{p})\cdot\mathbf{x} = 0$ (the tax raises zero revenue)

No Subsidies - in order to ensure positive tax revenue the government could set $q_i > p_i$ (i.e., $t_i > 0$) if $x_i > 0$ and $q_i < p_i$ (i.e., $t_i < 0$) if $x_i < 0$

Good - raises positive revenue

Bad - this tax is distortionary

Normalizing - if we normalize $p_1 = q_1 = 1$ we get $t_1 = 0$; if we do $\hat{p}_1 = 5$ and $\hat{q}_1 = 1$ we get $\hat{t}_1 = 4$ ... this shifts all taxes up by a factor of 4 (a proportional tax) so it doesn't change revenue) $\therefore$ original normalization with $t_1 = 0$ is OK

Welfare Maximization Problem - we're doing optimal taxation, but we don't look at government setting taxes ($\mathbf{t} = \mathbf{q} - \mathbf{p}$) because prices aren't tied down so we can get a multiplicity problem; instead we focus on setting prices (except $p_1 = q_1 = 0$)

$$\max_{\substack{q_2,\ldots,q_n \\ p_2,\ldots,p_n \\ z_1,\ldots,z_n}} V(\mathbf{q}) \quad \text{s.t.} \quad \begin{array}{l} \text{(i) } x_i(\mathbf{q}) - y_i - z_i = 0,\ i = 1,\ldots,n \quad \text{(market clearing)} \\ \text{(ii) } \mathbf{y} = \arg\max_{\mathbf{p}} \mathbf{p}\cdot\mathbf{y} \text{ s.t. } y_1 = f(y_2,\ldots,y_n) \\ \text{(iii) } z_1 = g(z_2,\ldots,z_n) \end{array}$$

Simplification - $x_i(\mathbf{q})$ is consumer demand (solution to the consumer maximization problem on top of page 2); we can't do the same for $\mathbf{y}$ because of constant returns to scale ($\mathbf{y}(\mathbf{p})$ would be unbounded), but we can play with the constraints to simplify the problem

Price Vector - $\mathbf{p}$ only enters in constraint (ii) so if we can drop this constraint, we don't have to worry about $\mathbf{p}$ (for now)

- First step is to use the market clearing condition (constraint (i)):

$$y_j = x_j(\mathbf{q}) - z_j,\ j = 2,\ldots,n$$

(note we didn't use $j = 1$ because we already normalized $p_1 = 0$)

- Next sub the production constraints ((ii) and (iii)) into the market clearing condition for good 1:

$$x_1(\mathbf{q}) = y_1 + z_1 = f(y_2,\ldots,y_n) + g(z_2,\ldots,z_n)$$

- Now sub the $y_2,\ldots,y_n$ we found in the first step:

$$x_1(\mathbf{q}) = f(x_2(\mathbf{q}) - z_2,\ldots,x_n(\mathbf{q}) - z_n) + g(z_2,\ldots,z_n)$$

New problem -

$$\max_{\substack{q_2,\ldots,q_n \\ z_2,\ldots,z_n}} V(\mathbf{q}) \quad \text{s.t.} \quad x_1(\mathbf{q}) = f(x_2(\mathbf{q}) - z_2,\ldots,x_n(\mathbf{q}) - z_n) + g(z_2,\ldots,z_n)$$

Note 1: z_1 is no longer a decision since $z_1 = g(z_2,\ldots,z_n)$

Note 2: we're using = constraint (not ≤) so we need to make sure second order conditions hold

Lagrangian - $L = V(\mathbf{q}) - \lambda\left[x_1(\mathbf{q}) - f(x_2(\mathbf{q}) - z_2,\ldots,x_n(\mathbf{q}) - z_n) - g(z_2,\ldots,z_n)\right]$

FOC - simplified notation: $\dfrac{\partial V}{\partial q_k} = V_k$; $\dfrac{\partial f}{\partial y_k} = f_k$

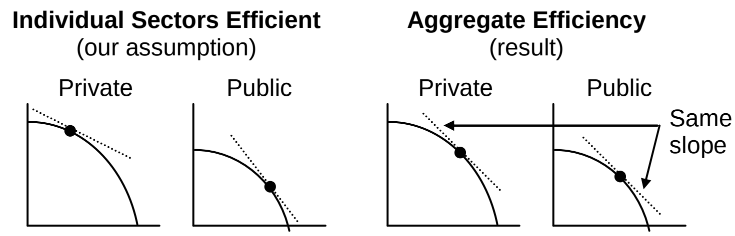
$$\mathbf{q}: \ \frac{\partial L}{\partial q_k} = V_k - \lambda\left[\frac{\partial x_1}{\partial q_k} - \sum_{i=2}^{n} f_i \frac{\partial x_i}{\partial q_k}\right] = 0,\quad k = 2,\ldots,n$$

$$\mathbf{z}: \ \frac{\partial L}{\partial z_k} = -\lambda\left[f_k - g_k\right] = 0,\quad k = 2,\ldots,n$$

Aggregate Production Efficiency - if it's an interior solution (i.e., $\lambda \neq 0$), the $\partial L / \partial z_k$ FOCs imply $f_k = g_k$... that means $\text{MRTS}^{\text{P}} = \text{MRTS}^{\text{G}}$ (marginal rates of technical substitution in private and public [government] sectors are equal; could also think of it as $\text{MRT}^{\text{P}} = \text{MRT}^{\text{G}}$ (marginal rates of transformation)

Result - with aggregate production efficiency, we can't reallocate $\mathbf{y}$ or $\mathbf{z}$ to produce more or produce same amount more efficiently

Graphs - we assumed production efficiency for private sector and public sector; that means each sector is operating on the frontier of it's PPF; for aggregate production efficiency, they're at points on their PPF that have the same slope



Interpretations - 3 ways to look at aggregate production efficiency

(1) **No Intermediate Goods Tax** - if we disaggregate private production sector we need a price vector for each sector so we couldn't use $\mathbf{y}(\mathbf{p})$ like we did, but aggregate production efficiency means all price vectors should be equal; that means business to business transactions are untaxed

Transactions -

Business to business	untaxed for aggregate production efficiency
Business to consumer	
Business to government	untaxed for aggregate production efficiency
Consumer to government	
Consumer to consumer	untaxed (both face same prices, $\mathbf{q}$)

Note: this implies government charges different price to consumers and businesses; other models don't allow this so they don't end up with aggregate production efficiency

Don't Need Intermediate Tax - choosing $\mathbf{q}$ ties down $\mathbf{x}(\mathbf{q})$ so we don't need an intermediate tax to control consumption

(2) **Untaxable Sectors** - subsistence agriculture (people who grow own food and eat it) or household production (home schooling, laundry, cooking, etc.) all gets lumped into consumer sector, but this model is focused on transactions (net trades), not final consumption so these activities are not taxed

(3) **Const-Benefit Analysis** - don't have to confine interpretation to static model; Arrow & Debrew view it as dynamic (just relabel commodity name for time periods); have to worry about discount rate

Tax on Interest Income - consumer sees $r(1-t)$ and producer sees r;

Aggregate production efficiency says government project should use producer discount rate

Interior Solution - what's required to guarantee $\lambda \neq 0$ (i.e., we have aggregate production efficiency)? Look at in terms of many consumer economy so rather than using $V(\mathbf{q})$ (derived from $\mathbf{x}(\mathbf{q})$) a modification of a Samuelson social welfare function

SWF - $W(\mathbf{x}) = W(u^1(\mathbf{x}^1), u^2(\mathbf{x}^2), \ldots, u^H(\mathbf{x}^H))$

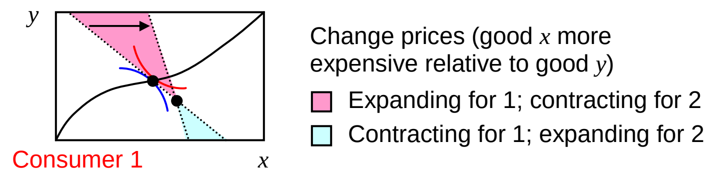
Modified - sub consumer demands: $V(\mathbf{q}) = W(u^1(\mathbf{x}^1(\mathbf{q})), u^2(\mathbf{x}^2(\mathbf{q})), \ldots, u^H(\mathbf{x}^H(\mathbf{q})))$

Note: this is assuming individualistic SWF; otherwise we'd have

$V(\mathbf{q}) = B(\mathbf{x}^1(\mathbf{q}), \mathbf{x}^2(\mathbf{q}), \ldots, \mathbf{x}^H(\mathbf{q}))$ ... which is not based on utilities

Individual Consumer - showed production will be at last intersection of offer curve and PPF so we know $\lambda \neq 0$ (see top of page 2)

Multiple Consumers - change in prices can bring gains to some consumers and losses to others (based on consumption possibilities) $\therefore$ we're not guaranteed to have an optimum that has aggregate production efficiency (i.e., could have $\lambda = 0$)



Aggregate Offer Curve - $X(\mathbf{q}) = \sum_{i=1}^{H} \mathbf{x}^i(\mathbf{q})$

Guarantee Improvement - want to find conditions under which $\Delta\mathbf{q}$ makes everyone better off so we'll have $\lambda \neq 0$ (aggregate production efficiency)

<u>Assume</u>... individualistic SWF: $V(\mathbf{q}) = W(u^1(\mathbf{x}^1(\mathbf{q})), u^2(\mathbf{x}^2(\mathbf{q})), \ldots, u^H(\mathbf{x}^H(\mathbf{q})))$

<u>Assume</u>... $\exists$ good j such that $x_j^h \leq 0 \;\forall\; h$ and $x_j^h < 0$ for some h (i.e., some consumers are net sellers of good j and others don't trade good j); could also use j such that $x_j^h \geq 0 \;\forall\; h$ and $x_j^h > 0$ for some h (i.e., some consumers are net buyers of good j and others don't trade good j); important thing is to have all consumers on <u>same side of market</u> for one good

Sub individual's indirect utility functions into SWF:

$V(\mathbf{q}) = W(v^1(\mathbf{q}), v^2(\mathbf{q}), \ldots, v^H(\mathbf{q}))$

How does price change affect social welfare? Take derivative:

$$\frac{\partial V}{\partial q_j} = \sum_{h=1}^{H} \frac{\partial V}{\partial u^h} \frac{\partial u^h}{\partial x^h} \frac{\partial x^h}{\partial q_j} = \sum_{h=1}^{H} V_h(-\alpha^h x_j^h)$$

Note: "individualistic" means $\frac{\partial V}{\partial q_i} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial q_i} = -\alpha x_i$ (see bottom of page 2)

We know $V_h = \frac{\partial V}{\partial u^h} > 0$ (make 1 consumer better off and [all else equal] social welfare improves)

Also know $\alpha^h > 0$ (marginal utility of income)

Sign of x_j^h depends $\therefore$ if all $x_j^h \geq 0 \Rightarrow \frac{\partial V}{\partial q_j} < 0$; or if all $x_j^h \leq 0 \Rightarrow \frac{\partial V}{\partial q_j} > 0$

i.e., if all consumers are on same side of market for some good, we can make all consumers better off (hence raise social welfare) by changing price $\therefore$ everyone is made better off by moving to the last intersection of aggregate offer curve and PPF (i.e. aggregate production efficiency)

Labor - we assume this condition holds for labor (all consumers are suppliers); but this doesn't work if we categorize labor (e.g., high and low skill)

Tax Rules - observations of the optimal commodity tax (by torturing the FOCs)

Go back to FOC for q_k (from bottom of p.4): $\dfrac{\partial L}{\partial q_k} = V_k - \lambda\left[\dfrac{\partial x_1}{\partial q_k} - \sum_{i=2}^{n} f_i \dfrac{\partial x_i}{\partial q_k}\right] = 0$, $k = 2,\ldots,n$

We know $p_i = -p_1 f_i \ \forall \ i = 2,\ldots,n$ (comes from private producer max profit; top of p.3)

Since good 1 is numeraire $p_1 = 1 \ \therefore \ p_i = f_i \ \forall \ i = 2,\ldots,n$

Sub that into the FOC for q_k: $V_k - \lambda\left[\dfrac{\partial x_1}{\partial q_k} + \sum_{i=2}^{n} p_i \dfrac{\partial x_i}{\partial q_k}\right] = V_k - \lambda \sum_{i=1}^{n} p_i \dfrac{\partial x_i}{\partial q_k} = 0$, $k = 2,\ldots,n$

We know $q_i = p_i + t_i$; gov't sets both $\mathbf{p}$ and $\mathbf{q}$ so $\dfrac{\partial x_i}{\partial q_k} = \dfrac{\partial x_i}{\partial t_k}$

Sub that into the FOC for q_k: $V_k - \lambda \sum_{i=1}^{n} p_i \dfrac{\partial x_i}{\partial t_k} = 0$, $k = 2,\ldots,n$

Derivative is linear operator so: $V_k = \lambda \dfrac{\partial}{\partial t_k}\left[\sum_{i=1}^{n} p_i x_i\right]$, $k = 2,\ldots,n$

Go back to tax: $\mathbf{t} = \mathbf{q} - \mathbf{p} \ \Rightarrow \ \mathbf{t}\cdot\mathbf{x} = \mathbf{q}\cdot\mathbf{x} - \mathbf{p}\cdot\mathbf{x}$, but $\mathbf{q}\cdot\mathbf{x} = 0$ (middle of p.1) $\therefore \ \mathbf{t}\cdot\mathbf{x} = -\mathbf{p}\cdot\mathbf{x}$

Sub that in: $V_k = -\lambda \dfrac{\partial}{\partial t_k}\left[\sum_{i=1}^{n} t_i x_i\right]$, $k = 2,\ldots,n$

(1) ΔTax Revenue > 0 $\Rightarrow$ ΔWelfare < 0:

$V_k =$ marginal utility to society of price change of good k; for 1 consumer case, $V_k = -\alpha x_k$ (bottom of p.2); that means, if consumer is net buyer ($x_k > 0$), raising the price lowers social welfare ($V_k < 0$)

$\dfrac{\partial}{\partial t_k}\left[\sum_{i=1}^{n} t_i x_i\right] =$ marginal tax revenue with respect to t_k (i.e., how total tax revenue changes based on a change in the tax on good k; it's not as simple to compute as just $\Delta t_k \Delta x_k$ because changing the price of good k, potentially changes the amount consumed of other goods

For the case where $V_k < 0$ (social welfare declines from raising price on good for which consumer is net buyer), we must have marginal tax revenue > 0 (i.e., positive tax revenue must be raised in order to make up for hurting consumers)

(2) Marginal Tax Revenue vs. Consumption:

For one consumer case $V_k = -\alpha x_k = -\lambda \dfrac{\partial}{\partial t_k}\mathbf{t}\cdot\mathbf{x} \ \Rightarrow \ x_k = \dfrac{\lambda}{\alpha}\dfrac{\partial}{\partial t_k}\mathbf{t}\cdot\mathbf{x}$, $k = 2,\ldots,n$

That says marginal tax revenue wrt t_k (tax on good k) is proportional to consumer's consumption of good k

(3) These Also Hold for Numeraire (Good 1)

The previous rules were for $k = 2,\ldots,n$, but they also hold for $k = 1$

Go back to using p_i instead of t_i in the FOC: $V_k = \lambda \dfrac{\partial}{\partial t_k}\left[\sum_{i=1}^{n} p_i x_i\right]$, $k = 2,\ldots,n$

V is homogeneous of degree 0 in $\mathbf{q}$: $\sum_{k=1}^{n} q_k V_k = 0$ (changing all prices by the same amount doesn't change social welfare)

Demand is homogeneous of degree 0 in $\mathbf{q}$: $\sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} q_k = 0$

Get tricky: multiply the demand condition by $\lambda \sum_{i=1}^{n} p_i$ (a constant times zero still equals zero):

$$\lambda \sum_{i=1}^{n} p_i \left(\sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} q_k \right) = 0$$

Add this to the first homogeneity condition: $\sum_{k=1}^{n} q_k V_k - \lambda \sum_{i=1}^{n} p_i \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} q_k = 0$

Do some fancy rearranging: $\sum_{k=1}^{n} \left[V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} \right] q_k = 0$

Return to FOC of q_k: $V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} = 0$, $k = 2, \ldots, n$ (version from the top of p.7)

We can multiple both sides by q_k: $\left[V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} \right] q_k = 0$, $k = 2, \ldots, n$

Add these up: $\sum_{k=2}^{n} \left[V_k - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_k} \right] q_k = 0$

Combine this with the sum from 1 to n (4 lines up) and we must have $V_1 - \lambda \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial q_1} = 0$

(4) Ramsey Rule - compensated demands for all goods change in <u>equal proportion</u> to the consumption of the goods; a tax that raises revenue causes fewer net trades

Consider 1 consumer economy with individualistic SWF
Gov't objective is $\max_{\mathbf{q}} V(\mathbf{q})$ (consumer's indirect utility function)

$V_k = -\alpha x_k = -\lambda \frac{\partial}{\partial t_k} \sum_{i=1}^{n} t_i x_i$ (middle of p.7, near "ΔTax Revenue > 0 $\Rightarrow$ ΔWelfare < 0")

$$\frac{\partial t_i x_i}{\partial t_k} = \begin{cases} x_i + t_i \frac{\partial x_i}{\partial t_k} & i = k \\ t_i \frac{\partial x_i}{\partial t_k} & i \neq k \end{cases}$$

$$\therefore\ -\alpha x_k = -\lambda \left(x_k + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial t_k} \right)$$

Slutsky Equation - $\frac{\partial x_i}{\partial q_k} = S_{ik} - x_k \frac{\partial x_i}{\partial I}$, where $S_{ik} = \frac{\partial x_i^{\mathrm{c}}}{\partial q_k}$

Recall $\frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial t_k}$ (top of p.7), so we can sub the Slutsky equation

$$-\alpha x_k = -\lambda\left(x_k + \sum_{i=1}^{n} t_i\left(S_{ik} - x_k \frac{\partial x_i}{\partial I}\right)\right)$$

$$-\alpha x_k = -\lambda x_k - \lambda\sum_{i=1}^{n} t_i S_{ik} + \lambda x_k \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial I}$$

$$\lambda\sum_{i=1}^{n} t_i S_{ik} = \left(\alpha - \lambda + \lambda\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial I}\right)x_k$$

$$\frac{\sum_{i=1}^{n} t_i S_{ik}}{x_k} = \frac{\alpha}{\lambda} - 1 + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial I}$$ ... note the right hand side is independent of k

Define $-\theta = \frac{\sum_{i=1}^{n} t_i S_{ik}}{x_k} = \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$ (although defined this way θ is independent of k)

($S_{ik} = S_{ki}$ because they are elements of the matrix of derivatives of compensated demand; $\mathbf{S}$ is symmetric and negative semidefinite)

Multiply both sides by $\sum_{k=1}^{n} t_k x_k$: $\sum_{k=1}^{n} t_k x_k \theta = -\sum_{k=1}^{n} t_k \cancel{x_k} \frac{\sum_{i=1}^{n} t_i S_{ki}}{\cancel{x_k}}$

$$\theta\sum_{k=1}^{n} t_k x_k = -\sum_{k=1}^{n}\sum_{i=1}^{n} t_k t_i S_{ki} = -\mathbf{t'St}$$

net tax revenue

This is quadratic form; $\mathbf{S}$ is negative semidefinite ($\mathbf{t'St} \le 0$) so $-\mathbf{t'St} \ge 0$

$\theta\sum_{k=1}^{n} t_k x_k \ge 0 \;\Rightarrow\;$ θ has same sign as net tax revenue

Back to $-\theta = \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$; Sub $S_{ki} = \frac{\partial x_k^c}{\partial q_i}$

$$\theta = \frac{\sum_{i=1}^{n} t_i \frac{\partial x_k^c}{\partial q_i}}{-x_k}$$

Gradient Approximation - t_i is the price (tax) change; $\partial x_k^c / \partial q_i$ is the change in compensated demand for good k from the price change on good i; if we assume $\mathbf{p}$ is constant, the numerator approximates Δx_k^c (change in compensated demand for good k) due to all taxes (Note: the results below still hold if $\mathbf{p}$ varies)

Ramsey Rule - if there is positive net tax revenue for a tax change ($\theta > 0$), the change in compensated demand for good k due to the tax is proportional to $-x_k$

Less Trades - $x_k > 0$ if buying; $x_k < 0$ if selling; change in proportion to $-x_k$ means $|x_k|$ gets smaller; that means there are fewer trades (less buying and selling)

(5) Regular Demands and Income - regular demands for all goods change in proportion (not equal) to the consumption of the goods; as income elasticity rises, the distortion (change in demand) is greater

Go back to $-\theta = \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$

Reverse the Slutsky equation: $\frac{\partial x_k}{\partial q_i} = S_{ki} - x_i \frac{\partial x_k}{\partial I} \Rightarrow S_{ki} = \frac{\partial x_k}{\partial q_i} + x_i \frac{\partial x_k}{\partial I}$

$$-\theta = \frac{\sum_{i=1}^{n} t_i \frac{\partial x_k}{\partial q_i} + \frac{\partial x_k}{\partial I} \sum_{i=1}^{n} t_i x_i}{x_k}$$

$$\frac{\sum_{i=1}^{n} t_i \frac{\partial x_k}{\partial q_i}}{x_k} = -\theta - \frac{\frac{\partial x_k}{\partial I} \sum_{i=1}^{n} t_i x_i}{x_k}$$

Gradient Approximation - same as bottom of p.9; t_i is the price (tax) change; $\partial x_k / \partial q_i$ is the change in (regular) demand for good k from the price change on good i; the numerator of the left hand side approximates Δx_k (change in demand for good k) due to all taxes

Interpretations - similar to Ramsey Rule in that the change in regular demand is proportional to the amount demanded, but there are a variety of interpretations based on the specific demand structure (e.g., taxes, after tax prices, etc.) and is "usually messy"; quantities (regular demands) are distorted by the tax anyway so it's easier to use the Ramsey Rule

Income Elasticity - from the right side, we can see that Δx_k (left side) is larger if $(\partial x_k / I) / x_k$ is larger; if we multiply that term by I we have income elasticity $\therefore$ larger income elasticity implies larger distortion from tax (i.e., more impact on Δx_k)

(6a) Deviate from Proportional Tax Locally - Corlett & Hague considered a proportional tax on all goods except the numeraire and looked to see if there are any local improvements (similar to Dixit, but without a lump-sum tax); got the same result as Dixit (i.e., want to deviate away from a proportional tax)

(6b) Deviate from Proportional Tax in Optimal 2nd Best - Diamond & Mirrlees; no assumption about proportional tax

Consider 3 goods (1 being the numeraire so $t_1 = 0$)

Apply this to $-\theta = \frac{\sum_{i=1}^{n} t_i S_{ki}}{x_k}$:

$t_2 S_{22} + t_3 S_{23} = -\theta x_2$ and $t_2 S_{32} + t_3 S_{33} = -\theta x_3$

After lots of magical algebra the end result is:

$$\frac{t_2}{q_2} >,=,< \frac{t_3}{q_3} \text{ as } \sigma_{21} >,=,< \sigma_{31}$$

English - optimal tax rates (based on consumer prices) follow the same order as compensated demand elasticities (between the good and the numeraire)

Assume $x_1 < 0$, $x_2 > 0$, $x_3 > 0$ (i.e., good 1 is labor which consumer sells to buy goods 2 and 3); the optimal tax rate is higher on the good that is more complementary to leisure (x_1)... i.e., tax rates are not proportional to amount consumed

(7) Special Cases

(7a) Cobb-Douglas has Proportional Tax - Cobb-Douglas utility function <u>with endowment of only one good</u> will have proportional taxes on all but the numeraire

$$U(\mathbf{x}) = b_1 \ln(x_1 + \omega_1) + \sum_{i=2}^{n} b_i \ln x_i$$

This is a Cobb-Douglas utility function

To make life even easier, assume $\sum_{i=1}^{n} b_i = 1$ (constant utility of income)

(Results below still hold without the constant utility of income)

Endowment - ω_1 is amount of good 1 available (doesn't matter which good is the endowment as long as it's only one good)

Buying & Selling - note structure of utility function implies the consumer will sell the endowment to buy other goods:

(a) $x_1 + \omega_1 \geq 0 \Rightarrow$ sell good 1 so $x_1 < 0$, but can't sell more than you have, ω_1

(b) $x_i \geq 0 \Rightarrow$ goods $2, \ldots, n$ are what the consumer buys

Budget Constraint - $\mathbf{q} \cdot \mathbf{x} = \mathbf{q} \cdot \boldsymbol{\omega} = \omega_1$

Cobb-Douglas - know structure of optimal consumption:

$$x_j = \frac{b_j \omega_1}{q_j} \;\; (j = 2, \ldots n) \text{ and } x_1 = \frac{(b_1 - 1)\omega_1}{q_1} = (b_1 - 1)\omega_1$$

Uncompensated Elasticities -

$$\frac{\partial x_i}{\partial q_k} = 0 \text{ for } i \neq k \;\Rightarrow\; \varepsilon_{ik} = 0$$

$$\frac{\partial x_k}{\partial q_k} = \frac{\partial}{\partial q_k}\left(\frac{b_k \omega_1}{q_k}\right) = -\frac{b_k \omega_1}{q_k^2} = -\frac{b_k \omega_1}{q_k}\frac{1}{q_k} = -\frac{x_k}{q_k} \;\Rightarrow\; \frac{q_k}{x_k}\frac{\partial x_k}{\partial q_k} = \varepsilon_{kk} = -1 \;\; (k = 2, \ldots n)$$

FOC wrt q_k: $-\alpha x_k = -\lambda\left(x_k + \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial t_k}\right)$ (version used in Ramsey Rule, bottom of p.8)

Recall $\frac{\partial x_i}{\partial q_k} = \frac{\partial x_i}{\partial t_k}$ (top of p.7): $-\alpha x_k = -\lambda x_k - \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial q_k}$

Apply results of uncompensated elasticities: $-\alpha \cancel{x_k} = -\lambda \cancel{x_k} + \lambda t_k \frac{\cancel{x_k}}{q_k} \quad (k = 2, \ldots n)$

$$\frac{\lambda-\alpha}{\lambda}=\frac{t_k}{q_k} \quad (k=2,\ldots n)$$

So all goods (except numeraire) are taxed at the same rate; the tax is proportional

(7b) Inverse Elasticity Rule - if $\varepsilon_{ik}=0\ \forall\ i=1,\ldots,n$, $k\neq i$ (i.e. goods are not related), then the tax rate on good k is inversely proportional to ε_{kk}

FOC wrt q_k: $-\alpha x_k=-\lambda\frac{\partial}{\partial t_k}\sum_{i=1}^{n}t_i x_i$ (middle of p.7, "ΔTax Revenue > 0 $\Rightarrow$ ΔWelfare < 0")

Pull out the k th term: $-\alpha x_k=-\lambda x_k-\lambda\frac{\partial}{\partial t_k}\sum_{i=1}^{n}t_i x_i$

Move derivative into the summation; recall $\frac{\partial x_i}{\partial q_k}=\frac{\partial x_i}{\partial t_k}$: $-\alpha x_k=-\lambda x_k-\lambda\sum_{i=1}^{n}t_i\frac{\partial x_i}{\partial q_k}$

From $\varepsilon_{ik}=0\ \forall\ i\neq k \Rightarrow \frac{\partial x_i}{\partial q_k}=0\ \forall\ i\neq k$: $-\alpha x_k=-\lambda x_k-\lambda t_k\frac{\partial x_k}{\partial q_k}$

Multiply both sides by $\frac{q_k}{x_k}$: $\frac{q_k}{x_k}(\lambda-\alpha)x_k=-\lambda t_k\frac{\partial x_k}{\partial q_k}\frac{q_k}{x_k}$

Recognize $\varepsilon_{ik}=\frac{\partial x_k}{\partial q_k}\frac{q_k}{x_k}$: $q_k(\lambda-\alpha)=-\lambda t_k\varepsilon_{kk}$

$\frac{(\lambda-\alpha)}{\lambda}=-\frac{t_k}{q_k}\varepsilon_{kk}$ ... tax rate * uncompensated demand elasticity constant across goods

$\frac{t_k}{q_k}=-\frac{(\lambda-\alpha)}{\lambda}\frac{1}{\varepsilon_{kk}}$ ... tax rate is inversely proportional to elasticity

General Rule - result if approximation if own effects are much larger than cross effects

Cobb Douglas - original assumption is just $\varepsilon_{ik}=0\ \forall\ i=1,\ldots,n$, $k\neq i$, but if we look at budget constraint ($\mathbf{q}\cdot\mathbf{x}=0$) we get interesting result (it only happens with Cobb-Douglas preferences):

Differentiate wrt q_k: $x_k+\sum_{i=1}^{n}q_i\frac{\partial x_i}{\partial q_k}=0$

$\varepsilon_{ik}=0\ \forall\ i\neq k \Rightarrow \frac{\partial x_i}{\partial q_k}=0$, $i\neq k$: $x_k+q_k\frac{\partial x_k}{\partial q_k}=0$

Multiply both sides by $\frac{1}{x_k}$: $1+\frac{q_k}{x_k}\frac{\partial x_k}{\partial q_k}=0 \Rightarrow \frac{q_k}{x_k}\frac{\partial x_k}{\partial q_k}=\varepsilon_{kk}=-1$

Constant elasticity of demand results from Cobb-Douglas preferences

Demand Restriction - doesn't make sense that $\varepsilon_{1k}=0$ if good 1 has the endowment (i.e., consumer sells good 1 to buy all other goods); changes FOC above:

$x_k+q_1\frac{\partial x_1}{\partial q_k}+q_k\frac{\partial x_k}{\partial q_k}=0 \Rightarrow \frac{q_k}{x_k}\frac{\partial x_k}{\partial q_k}=\varepsilon_{kk}=-1-\frac{q_1}{x_k}\frac{\partial x_1}{\partial q_k}$

Complex - don't know what preferences result in these demand elasticities

Dangerous - asking numeraire (good 1) to do a lot

Don't Add Much - this rule applies to Cobb Douglas preferences ($\varepsilon_{ii} = -1 \;\forall\; i = 1,\ldots,n$) which agrees with rule 7a that tax rate is constant across all goods

Private Production - so fare we only considered having constant returns in private production sector; anything with increasing returns was put in government sector (to avoid regulation issue); we ignored decreasing returns; adding that to the private sector could result in profits

Profit Tax - if there's a 100% profits tax (on pure profit, not counting return to capital), the same optimal tax results go through

< 100% - papers by Munk; profits get returned to consumers in closed GE model so it changes the budget constraint: $\mathbf{q}\cdot\mathbf{x} = I = (1-\tau)\mathbf{p}\cdot\mathbf{y}$ (where τ is the profits tax rate)

Problem - in work above, we claimed $\mathbf{q}$ and $\mathbf{p}$ were independent (so we could do 2 normalizations); if $\mathbf{p}\cdot\mathbf{y} = 0$ (no profit like we assumed above) or $\tau = 1$ (100% tax) there's no problem, but otherwise we'll have $\mathbf{q}$ and $\mathbf{p}$ <u>not</u> independent

Consumer Demands - $\mathbf{x}(\mathbf{q}, I) = \mathbf{x}(\mathbf{q}, (1-\tau)\mathbf{p}\cdot\mathbf{y}) \;\Rightarrow\; \hat{\mathbf{x}}(\mathbf{q},\mathbf{p})$ ... depends on both consumer and producer prices; only get homogeneity of degree zero if we change $\mathbf{q}$ and $\mathbf{p}$ together

Solution- set p_1 so $I = 0$ (effectively a 100% profits tax)... this is not possible if there is some good that is not taxed

Complexities - need to worry about demand effects of taxes and producer responses (supply curves); Munk, Stiglitz, and Dasgupta cover "a lot of gory detail"

Other Restrictions - other types of tax restrictions covered by Munk, Stiglitz, and Dasgupta; "no clean results"

Multiconsumer Economy - all previous work used a 1 consumer economy; could model with multiple consumer by using indirect utility in social welfare function:

$$V(\mathbf{q}) = W(v^1(\mathbf{q}), v^2(\mathbf{q}), \ldots, v^H(\mathbf{q}))$$

$$\frac{\partial V}{\partial q_k} = \sum_{h=1}^{H} \frac{\partial W}{\partial v^h}\frac{\partial v^h}{\partial q_k} = \sum_{h=1}^{H} \frac{\partial W}{\partial v^h}\left(-\alpha^h x_k^h\right) = -\sum_{h=1}^{H} \beta^h x_k^h$$

where $\dfrac{\partial v^h}{\partial q_k} = -\alpha^h x_k^h$ (middle of p.2)

$\alpha^h =$ private marginal utility of "income" for person h; (say "income" because we're really measuring consumption in terms of good 1)

$x_k^h =$ person h's consumption of good k

$\beta^h = \alpha^h \dfrac{\partial W}{\partial v^h} =$ social MU of consumption by person h

Correlations - Diamond-Mirrless Eqn 77 is a "real mess"; with multiple consumers demand reductions at optimal taxes depend on more factors ("correlations"):

β^h, x_k^h - is consumption concentrated around different MU of consumption?

$\beta^h, \dfrac{\partial}{\partial I^h}\sum_{i=1}^{n} t_i x_i^h$ - how change in income affects amount of commodity tax paid

$\beta^h, \left(\sum_{i=1}^{n} t_i x_i^h\right)\dfrac{\partial x_k^h}{\partial I^h}$ - product of tax paid and change in demand for good k wrt income

Pole Subsidy - Diamond (1975) cleaned up results from the "correlations" paper by adding a policy rule (pole subsidy) allowing government to return some of the commodity tax (same amount, I , for everyone); changes budget constraint: $\mathbf{q}\cdot\mathbf{x}^h = I$

Commodity vs. Income Tax - a commodity tax with same rate on all goods and $\mathbf{q}\cdot\mathbf{x} = 0$ is similar to an income tax

Add Public Goods - $\mathbf{e}$ (can be single good or many; same amount consumed by all consumers)

Consumer Problem - $\max\limits_{\mathbf{x}^h} u^h(\mathbf{x}^h, \mathbf{e})$

Maximized Value - since $\mathbf{x}^h(\mathbf{q}, I, \mathbf{e})$, max value (indirect utility) is $v^h(\mathbf{q}, I, \mathbf{e})$

Aggregate Consumption - $\mathbf{x} = \sum_{h=1}^{H} \mathbf{x}^h$; $x_i = \sum_{h=1}^{H} x_i^h$

Production Constraint - $F(\mathbf{x}^h(\mathbf{q}, I, \mathbf{e}), \mathbf{e}) = 0$

Social Problem - $\max\limits_{\mathbf{q}, I, \mathbf{e}} W(v^1, v^2, \ldots, v^H)$ s.t. $F = 0$

Producer prices ($\mathbf{p}$) are hidden in constraint

Lagrangian - $L = W - \lambda F$

FOC wrt q_k - $\dfrac{\partial L}{\partial q_k} = \sum_{h=1}^{H} \dfrac{\partial W}{\partial v^h}\dfrac{\partial v^h}{\partial q_k} - \lambda \sum_{i=1}^{n} F_i \dfrac{\partial x_i}{\partial q_k} = 0$ (using $F_i = \dfrac{\partial F}{\partial x_i}$)

Tricks:

Normalizations - $q_1 = p_1 = 1 = F_1$

Private Production Objective - $p_i = p_1 F_i \;\; \forall \;\; i = 2, \ldots, n$ (top of p.3)

Aggregate consumption - $x_i = \sum_{h=1}^{H} x_i^h$

Right term becomes: $\lambda \sum_{h=1}^{H} \sum_{i=1}^{n} p_i \dfrac{\partial x_i^h}{\partial q_k}$

Social MU of Consumption - $\beta^h = \alpha^h \dfrac{\partial W}{\partial v^h}$ (from p.13)

$\dfrac{\partial v^h}{\partial q_k} = -\alpha^h x_k^h$ (middle of p.2)

Left term becomes: $\sum_{h=1}^{H} \dfrac{\beta^h}{\alpha^h}\left(-\alpha^h x_k^h\right)$

FOC becomes: $\sum_{h=1}^{H} \beta^h x_k^h = -\lambda \sum_{h=1}^{H} \sum_{i=1}^{n} p_i \dfrac{\partial x_i^h}{\partial q_k}$

Replace $p_i = q_i - t_i$: $\sum_{h=1}^{H} \beta^h x_k^h = -\lambda \sum_{h=1}^{H} \sum_{i=1}^{n} (q_i - t_i) \dfrac{\partial x_i^h}{\partial q_k}$

Multiply out the second term: $\sum_{h=1}^{H} \beta^h x_k^h = -\lambda \sum_{h=1}^{H} \left(\sum_{i=1}^{n} q_i \dfrac{\partial x_i^h}{\partial q_k} - \sum_{i=1}^{n} t_i \dfrac{\partial x_i^h}{\partial q_k} \right)$

Trick: differentiate budget constraint $\mathbf{q}\cdot\mathbf{x}^h = I$ wrt q_k : $x_k^h + \sum_{i=1}^{n} q_i \dfrac{\partial x_i^h}{\partial q_k} = 0$

$\sum_{i=1}^{n} q_i \dfrac{\partial x_i^h}{\partial q_k} = -x_k^h$ ... sub this into the FOC: $\sum_{h=1}^{H} \beta^h x_k^h = -\lambda \sum_{h=1}^{H} \left(-x_k^h - \sum_{i=1}^{n} t_i \dfrac{\partial x_i^h}{\partial q_k} \right)$

Slutsky Equation - $\frac{\partial x_i^h}{\partial q_k} = S_{ik}^h - x_k^h \frac{\partial x_i^h}{\partial I}$ ... sub this into the FOC:

$$\sum_{h=1}^{H} \beta^h x_k^h = \lambda \sum_{h=1}^{H} \left(x_k^h + \sum_{i=1}^{n} t_i \left(S_{ik}^h - x_k^h \frac{\partial x_i^h}{\partial I} \right) \right)$$

Move terms around: $\sum_{h=1}^{H} \beta^h x_k^h = \sum_{h=1}^{H} \lambda x_k^h + \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} t_i S_{ik}^h - \sum_{h=1}^{H} \lambda \sum_{i=1}^{n} x_k^h t_i \frac{\partial x_i^h}{\partial I}$

$$\sum_{h=1}^{H} \left(\beta^h + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I} - \lambda \right) x_k^h = \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} t_i S_{ik}^h$$

Social MU of Income to Consumer *h* - $\gamma^h \equiv \beta^h + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I}$

Give income to an individual; direct impact is social MU of consumption (β^h) plus social effect on government budget (i.e., tax revenue); the effect on the budget constraint equals the social value of relaxing the budget constraint (λ) times change in taxes paid by consumer's extra income; sometimes refer to γ^h as a reduction in cost of giving extra income to consumers (government gets some of it back in form of commodity tax revenue)

Now FOC becomes: $\sum_{h=1}^{H} (\gamma^h - \lambda) x_k^h = \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} t_i S_{ik}^h$

Right side is change in compensated demand for good k due to taxes (similar to what we had for Ramsey Rule (p.8), but here the left side is different) :

$$\sum_{h=1}^{H} (\gamma^h - \lambda) x_k^h = \lambda \Delta x_k^c$$

FOC wrt *I* - $\frac{\partial L}{\partial I} = \sum_{h=1}^{H} \frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial I} - \lambda \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial I} = 0$

Tricks:

Same normalizations, etc. from p. 14 makes the right term: $\lambda \sum_{h=1}^{H} \sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial I}$

For left term use $\beta^h = \alpha^h \frac{\partial W}{\partial v^h}$ and $\frac{\partial v^h}{\partial I} = \alpha^h$: $\sum_{h=1}^{H} \frac{\beta^h}{\alpha^h} (\alpha^h)$

FOC becomes: $\sum_{h=1}^{H} \frac{\beta^h}{\alpha^h} (\alpha^h) = \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} p_i \frac{\partial x_i^h}{\partial I}$

Replace $p_i = q_i - t_i$: $\sum_{h=1}^{H} \beta^h = \lambda \sum_{h=1}^{H} \sum_{i=1}^{n} \left(q_i \frac{\partial x_i^h}{\partial I} - t_i \frac{\partial x_i^h}{\partial I} \right)$

Differentiate budget constraint $\mathbf{q} \cdot \mathbf{x}^h = I$ wrt I : $\sum_{i=1}^{n} q_i \frac{\partial x_i^h}{\partial I} = 1$

FOC becomes: $\sum_{h=1}^{H} \beta^h = \lambda \sum_{h=1}^{H} 1 - \sum_{h=1}^{H} \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I}$

$$\sum_{h=1}^{H}\left(\beta^h + \lambda\sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I}\right) = \sum_{h=1}^{H}\gamma^h = \lambda H$$

(1) social benefit of giving everyone a dollar = social cost of giving everyone a dollar

(2) $\lambda = \frac{1}{H}\sum_{h=1}^{H}\gamma^h =$ mean of γ^h

Combine FOCs - from interpretation (2) above, $\sum_{h=1}^{H}(\gamma^h - \lambda) = 0 \ \therefore\ \bar{x}_k\sum_{h=1}^{H}(\gamma^h - \lambda) = 0$;

add this to the FOC wrt q_k: $\sum_{h=1}^{H}(\gamma^h - \lambda)x_k^h = \lambda\Delta x_k^{c}$ (from p.15)

$$\sum_{h=1}^{H}(\gamma^h - \lambda)x_k^h - \bar{x}_k\sum_{h=1}^{H}(\gamma^h - \lambda) = \lambda\Delta x_k^{c}$$

Factor: $\sum_{h=1}^{H}(\gamma^h - \lambda)(x_k^h - \bar{x}_k) = \lambda\Delta x_k^{c}$

The left side is similar to covariance (just missing the denominator)

Results -

(a) if $Cov(\gamma^h, x_k^h) > 0$ (good concentrated among people with high γ) $\Rightarrow$ want to increase consumption (based on compensated demand) of good k

(b) if $Cov(\gamma^h, x_k^h) < Cov(\gamma^h, x_j^h)$ $\Rightarrow$ reduce consumption of good k more than that of good j

Public Goods

- already added them in the previous section; this section will get the equivalent of the Samuelson rule in the presence of commodity taxes

$V(\mathbf{q}, I, \mathbf{e}) = W(v^1(\mathbf{q}, I, \mathbf{e}), v^2(\mathbf{q}, I, \mathbf{e}), \ldots, v^H(\mathbf{q}, I, \mathbf{e}))$

Recall from previous section:

$\beta^h =$ social MU of consumption by person h $\qquad \beta^h = \frac{\partial W}{\partial v^h}\frac{\partial v^h}{\partial q_k}$

$\gamma^h =$ social MU of income to consumer h $\qquad \gamma^h \equiv \beta^h + \lambda\sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I}$

Public Good - now we'll look at how expenditure on public goods affects welfare:

Directly - how change in consumer's utility from change in public good affects welfare

Indirectly - how tax revenues change based on change in consumer demands from change in public goods

Social Marginal Benefit of Public Good for Consumer *h* - $\delta^h \equiv \frac{\partial W}{\partial v^h}\frac{\partial v^h}{\partial \mathbf{e}} + \lambda\sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial \mathbf{e}}$

Lagrangian - $L = W - \lambda F$

FOC wrt e - $\frac{\partial L}{\partial \mathbf{e}} = \sum_{h=1}^{H}\frac{\partial W}{\partial v^h}\frac{\partial v^h}{\partial \mathbf{e}} - \lambda\sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial \mathbf{e}} - \lambda F_{\mathbf{e}} = 0$ (using $F_i = \frac{\partial F}{\partial x_i}$; $F_{\mathbf{e}} = \frac{\partial F}{\partial \mathbf{e}}$)

Use $F_i = p_i = q_i - t_i$: $\sum_{h=1}^{H}\frac{\partial W}{\partial v^h}\frac{\partial v^h}{\partial \mathbf{e}} - \lambda\sum_{i=1}^{n} q_i \frac{\partial x_i}{\partial \mathbf{e}} + \lambda\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial \mathbf{e}} = \lambda F_{\mathbf{e}}$

Now, change in public goods can change demands (x_i), but it has no effect on the budget constraint $\therefore \sum_{i=1}^{n} q_i \frac{\partial x_i}{\partial \mathbf{e}} = 0$

Can break out consumers so $\sum_{i=1}^{n} t_i \frac{\partial x_i}{\partial \mathbf{e}} = \sum_{h=1}^{H} \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial \mathbf{e}}$

FOC becomes: $\sum_{h=1}^{H} \left(\frac{\partial W}{\partial v^h} \frac{\partial v^h}{\partial \mathbf{e}} + \lambda \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial \mathbf{e}} \right) = \sum_{h=1}^{H} \delta^h = \lambda F_{\mathbf{e}}$

Samuelson Rule - says $\sum \mathrm{MRS} = \mathrm{MRT}$ for public goods

Modified - $\mathrm{MRT} = F_{\mathbf{e}} = \frac{1}{\lambda} \sum_{h=1}^{H} \delta^h$

From previous section $\lambda = \frac{1}{H} \sum_{h=1}^{H} \gamma^h$: $\mathrm{MRT} = F_{\mathbf{e}} = \frac{\sum_{h=1}^{H} \delta^h}{\sum_{h=1}^{H} \gamma^h} = \frac{\text{Sum of social gain from public good}}{\text{Sum of social gain from private good (income)}}$

Difference - Samuelson assumed lump sum tax paid for the public goods so $t_i = 0$; in this case, we can't combine terms; end up with MRT = weighted sum of MRS + "external gain" from public good (i.e., additional commodity taxes raised)

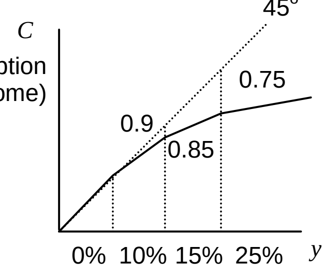
Optimal Income Taxation

Indirect Taxes - (e.g., taxes on commodities); government only interacts with consumers through markets (consumers are anonymous)
Direct Taxes - (e.g., income tax); government treats consumers as individuals
Issues -

(1) Distribution of burden of public expenditure
(2) Do we want to actively transfer income to some individuals (pole tax in commodity tax section returned tax equally to all consumers)
(3) Factors in income tax schedules (e.g., vary with family size)
(4) What should pattern of marginal tax rate be

Utilitarians - addressed first two issues using inelastic labor supply; argued for different tax rates based on ability to pay (with inelastic labor supply, income measures ability to pay)
Elastic Labor Supply - income tax creates labor supply distortions (so utilitarian results not very useful); ability to pay is determinant of income but so is effort

Effect of Increasing Marginal Tax Rates



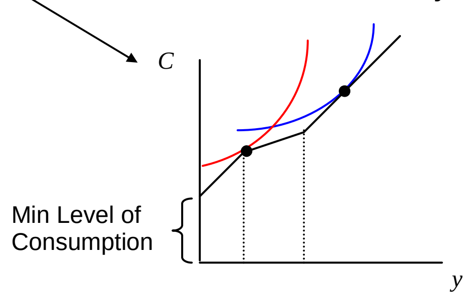
Linear Income Tax - $T = \begin{cases} 0 & \text{if } y \le \hat{y} \\ x\%(y - \hat{y}) & \text{if } y > \hat{y} \end{cases}$

Modify for Min Consumption - $T = -a + x\% y$

Real World - marginal tax rates vary with income
Lots o' Parameters - include break points, marginal tax rate at each break point, number of break points
Simple Say - fully non-linear income tax

Vickrey - first thought about optimal income tax problem, but couldn't solve it because choices depend on entire budget set (not convex)

High Tax Rate on EIT Induce Non-Convexity



Asymmetric Information Problem - Mirrlees insight; income is choice which depends on ability, effort and tax schedule; government only observes income
Ability - wage; fixed in short-run (i.e., ability and effort are not related); Mirrlees used continuum of abilities, but finite case is easier to understand
Effort - hours worked
Income - wage times hours
Revelation Principle - modeling strategy; have people report wage and assign them effort level and tax (i.e., income) based on wage (this doesn't make sense because government can't observe wage but it's just the model to derive the tax schedule for the real problem: people report incomes)
Implied Tax Schedule - if presented with tax schedule developed this way, people will make same choice of effort when they report income rather than wage
Incentive Compatibility - want to guarantee people will tell the truth
Brito & Oakland - solve problem with continuum of abilities using optimal control (easier than the way Mirrlees solved it)

Stiglitz - used finite types of ability (wages): w_1 & w_2

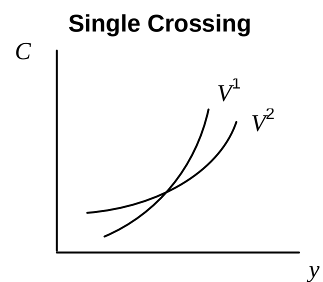
Income - $y_i = w_i L_i$ ($L_i =$ labor [effort])

Ability - government doesn't know an individual's type, but does know distribution of types; N_1 people of type w_1

Consumer Utility - $U^i(c_i, L_i)$ (usually assume it's concave);

$U_c > 0$ (consumption good) and $U_L < 0$ (labor bad)

Budget Constraint - $c_i = y_i - T(y_i)$



Observable Utility - $U^i(c_i, L_i) = U^i(c_i, y_i / w_i)$, but government doesn't observe labor (effort) or wage (ability) so embed the wage into the function (which isn't actually observed anyway) and use $V^i(c_i, y_i)$

Single Crossing Assumption - indifference curves cross at most once; at any point of intersection (low ability) V^1's indifference curve is steeper than V^2's

Common Utility Function - if $U(c, L)$ is identical for each consumer so only wage (ability) differences change the slope of the indifference curves (recall we could also write it as $U(c, y/w)$), then single crossing results from consumption of normal good; common utility function is not required for most of the results we'll see, but it helps for intuition

Tax System - defines relationship between consumption and earned income (see budget constraint above)

Taxes Paid - type i consumer pays $y_i - c_i$

Direct Mechanism - based on wage; can't actually implement as a tax system, but uses revelation principle to derive tax schedule; individual reports ability (wage) and government assigns that person an income (hence effort) and tax (hence consumption)

Gov't Goal - want type w_i to consume c_i and earns y_i

Choice - person with w_2 has two options:

(1) Report w_2 and get $V^2(c_2, y_2)$

(2) Report w_1 and get $V^2(c_1, y_1)$

Incentive Compatibility (IC) - also called **self-selection**; to get person with w_2 to tell the truth, ensure: $V^2(c_2, y_2) \geq V^2(c_1, y_1)$

For person with w_1: $V^1(c_1, y_1) \geq V^1(c_2, y_2)$ (usually not binding)

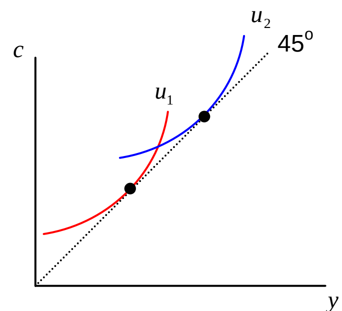
Pareto Problem - this is how Stiglitz solved the problem

$$\max_{c_1, y_1, c_2, y_2} V^2(c_2, y_2)$$

s.t. $V^1(c_1, y_1) \geq \bar{u}^1$	(μ) (Pareto constraint; type 1's utility)
$N_1(y_1 - c_1) + N_2(y_2 - c_2) \geq \overline{R}$	(γ) (Gov't revenue requirement [budget constraint])
$V^2(c_2, y_2) \geq V^2(c_1, y_1)$	(λ_2) (IC for type 2)
$V^1(c_1, y_1) \geq V^1(c_2, y_2)$	(λ_1) (IC for type 1)

Binding Constraints - Stiglitz assumed μ and γ > 0 (i.e., Pareto and revenue constraints are binding); then he considered cases for the IC constraints

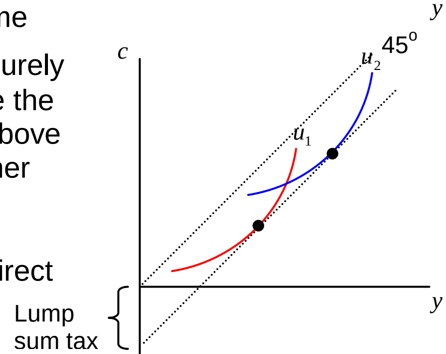
Graphs - consider competitive equilibrium (no tax); both types are on the 45° line so $\text{MRS}^i = -\dfrac{\partial V^i/\partial y}{\partial V^i/\partial c} = 1$ ($i = 1,2$) at equilibrium points $\therefore$ no distortion in labor supply



Lump Sum Tax - shifts consumer budget line down (less income); still have $\text{MRS}^i = 1$ so there's no distortion in labor supply

Assumptions - from here on out we'll assume $N_1 = N_2$ (same number of each type of consumer) and $\overline{R} = 0$ so tax is purely redistributive... makes graphs easier to interpret because the amount of transfer (shift in one consumer's budget line above the 45° line) has to be offset by lump sum tax (shift in other consumer's budget line below the 45° line)

"Allocation" - government assigns income & tax... hence consumption so consider (y, c) being assigned for the direct mechanism... just using this to design the tax schedule



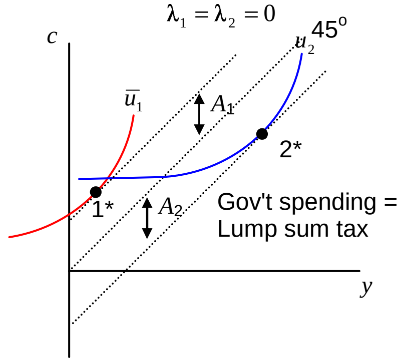
Back to Stiglitz's three cases:

(1) **Neither Binds** - $\lambda_1 = \lambda_2 = 0$; self selection constraints are satisfied; each allocation lies below the other type's indifference curve $\therefore$ for small amounts of redistribution there is no distortion

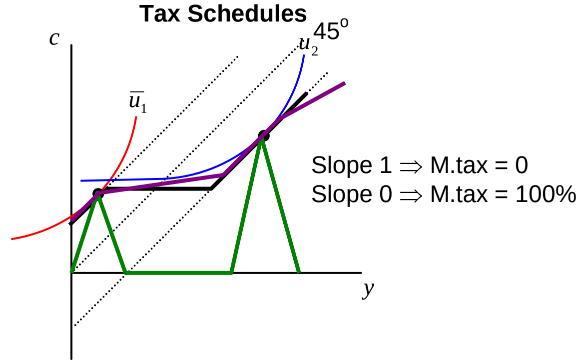
- **Marginal Tax Rate** - $1 - \text{MRS}^i$ at i's bundle
- **Lump Sum Tax** - in this case the income tax schedules look like a lump sum tax; each type either gets benefit A_1 or tax A_2, both at 0% marginal tax rate
  - **Result** - marginal tax rate will be zero if IC constraints don't bind
- **Tax Schedule** - requirements:
  - (a) has to go through 1* and 2*
  - (b) has to be tangent to indifference curves at 1* and 2*
  - (c) has to stay below the indifference curves through 1* and 2* everywhere else
  - **Not Unique** - as long as these three criteria are met, the tax schedule can be anything

Self Selection Satisfied

$\lambda_1 = \lambda_2 = 0$



Tax Schedules



(2) **High Ability Binds** - $\lambda_1 = 0$ & $\lambda_2 > 0$; in first graph below, both consumers prefer 1* so self selection constraint for type 2 is violated; we can't move type 1's indifference curve because of Pareto constraints (already at $\bar{u}_1$) $\therefore$ we have to slide type 1's allocation along this indifference curve... that requires extra revenue so we have to

move type 2 to a lower indifference curve; to maximize R (i.e., shift 2 down the minimum amount) we want to keep the type 2 allocation with MRS = 1 (zero marginal tax rate)

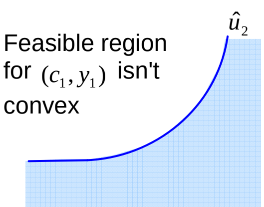
Tax Schedule - not differentiable; has to be tangent to $\bar{u}_1$ from the left and tangent to $\hat{u}_2$ from the right

Result - at final allocation for type 1: $MRS^2 > MRS^1 > 1$ (i.e., marginal tax rate for type 1 is > 0%; there is labor distortion and type 1 works less than he should); at allocation for type 2: $MRS^2 = 1$ (zero marginal tax for type 2)

Source of Distortions -

Type 1 - $V^1(\underbrace{c_1, y_1}) \geq \bar{u}^1$ and $V^2(c_2, y_2) \geq V^2(\underbrace{c_1, y_1})$

Want (c_1, y_1) big for Pareto const. but small for 2's IC const... <u>conflict</u>

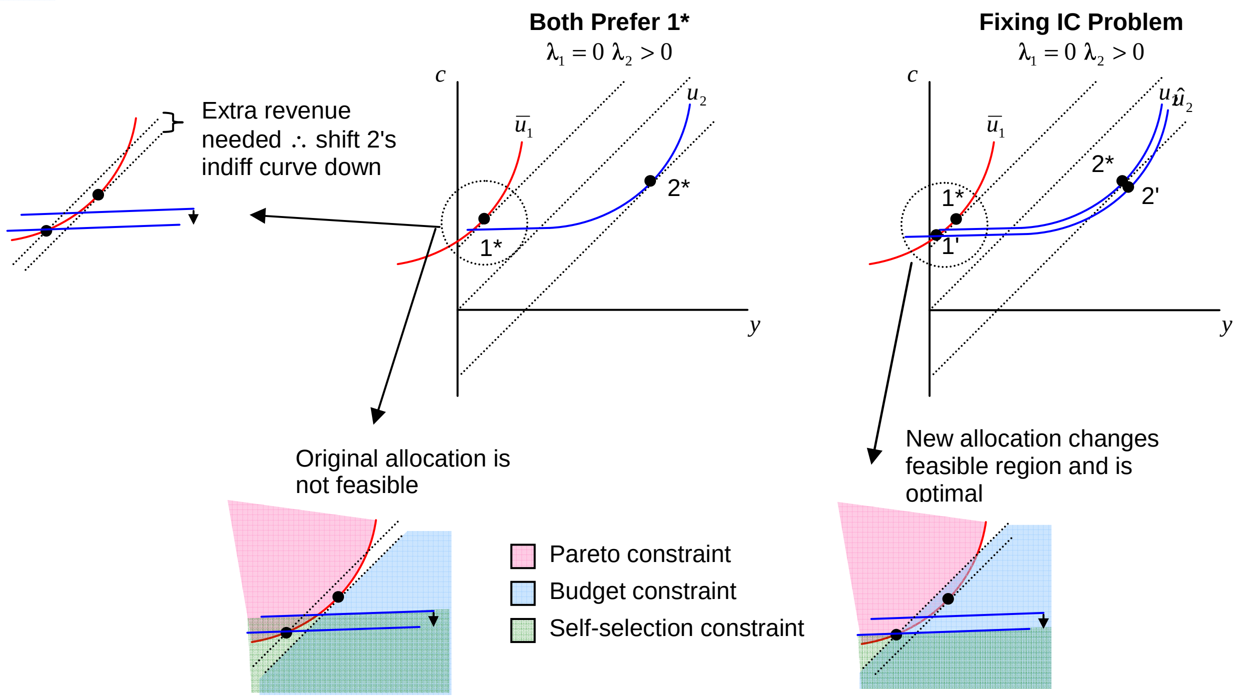


Feasible region for (c_1, y_1) isn't convex

Type 2 - $\max\limits_{c_1, y_1, c_2, y_2} V^2(\underbrace{c_2, y_2})$ and $V^2(\underbrace{c_2, y_2}) \geq V^2(c_1, y_1)$

Want to make both bigger... no conflict

Problem - once 2's allocation is fixed, we have to move (c_1, y_1) on or below $\hat{u}_2$ $\therefore$ $V^2(c_2, y_2) \geq V^2(c_1, y_1)$ defines a non-convex set



(3) **Low Ability Binds** - $\lambda_1 > 0$ & $\lambda_2 = 0$; in first graph below there's no problem; all we did was switch who's paying the tax and who's receiving the transfer (normally we'd think of the high wage paying to support the low wage, but that's not required... just play along); in the second graph, both consumers prefer 2* so self selection constraint for type 1 is violated; we're not going to move type 1's indifference curve again, instead we'll lower type 2's indifference curve to the point where it intersects type 1's at the budget line (i.e., no change in the lump sum transfer)

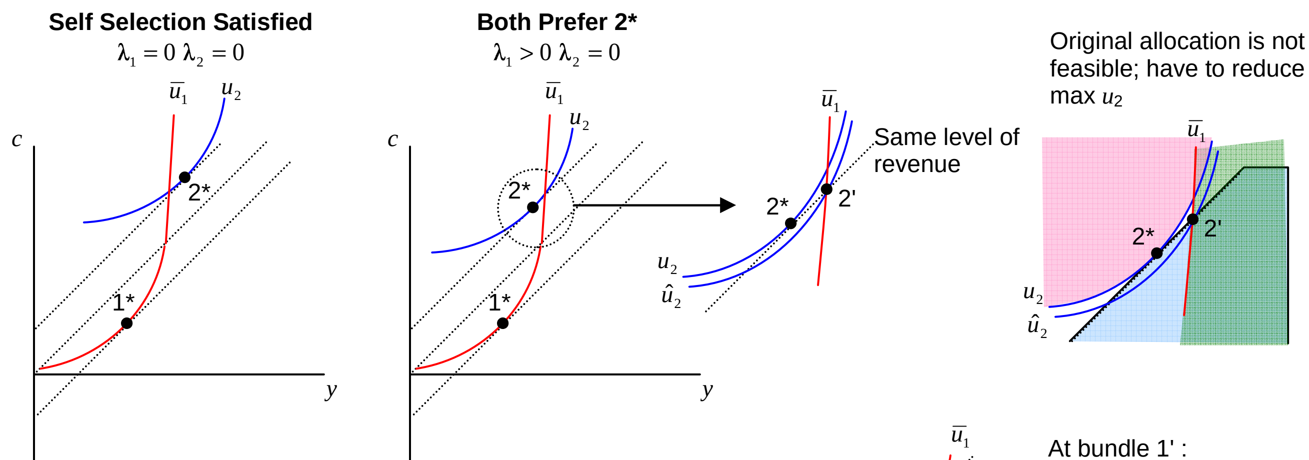
Tax Schedule - as before it's not differentiable; has to be tangent to $\bar{u}_1$ from the left and tangent to $\hat{u}_2$ from the right (same as before)

Result - at final allocation for type 2: MRS[1] > MRS[2] > 1 (i.e., marginal tax rate for type 2 is <0%; there is labor distortion and type 2 works more than he should); at allocation for type 1: MRS[1] = 1 (zero marginal tax for type 1)

Source of Distortions -

$$\max_{c_1,y_1,c_2,y_2} V^2(c_2,y_2) \text{ and } V^2(c_2,y_2) \geq V^2\underbrace{(c_1,y_1)} \text{ and } V^2(c_2,y_2) \geq V^2\underbrace{(c_1,y_1)}$$

Want to make (c_2,y_2) "smaller" so type 1 doesn't like it, but do so at min utility cost to type 2 because objective requires (c_2,y_2) "bigger"



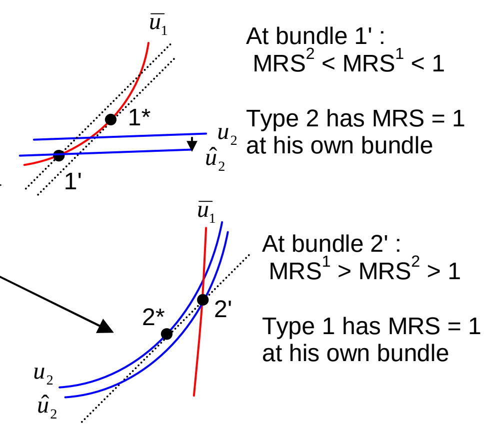
Stiglitz Summary - 2 goods (consumption & labor) and 2 types (high and low ability)

Type 2 Self Selection - $V^2(c_2,y_2) \geq V^2(c_1,y_1)$ ... if binding, adjust (c_1,y_1) so MRS[1] < 1 (distort labor supply for type 1)

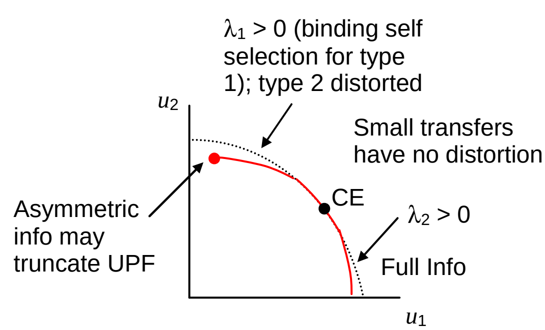
Type 1 Self Selection - $V^1(c_1,y_1) \geq V^1(c_2,y_2)$ ... if binding, adjust (c_2,y_2) so MRS[2] > 1 (distort labor supply for type 2)

No Distortion - for small amounts of redistribution

Single Crossing - assumed indifference curves have this property; used this to derive fact that at most 1 self-selection constraint binds, which then implicitly suggested type paying greatest total tax is not envied by anyone

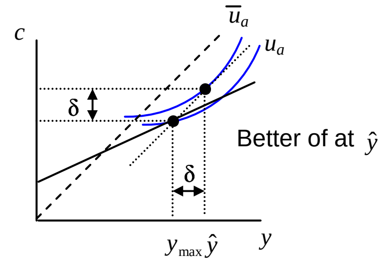


Utility Possibilities Frontier - asymmetric information causes distortions the further away from competitive equilibrium; may even have truncated UPF so some values are not attainable; full information case (self selection constraints irrelevant) is the dotted line

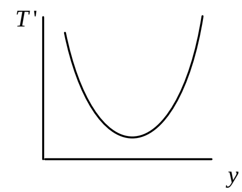


Continuous Ability Distribution -
Sadka - proposed theorems:
- (Mirrlees) If tax schedule is optimal under additive social welfare function, then it is <u>nondecreasing</u> (i.e., the marginal rate is non-negative)
- <u>Zero marginal tax</u> rate for top income person... look at graph; if top income person is at $y_{\max}$ with a positive marginal rate, it's possible to get him to produce more income ($\hat{y}$), make him better off (higher utility), and keep government revenue constant by giving him a zero marginal tax rate (MRS = 1)

  Limited Applicability - only works for the top income earner



Diamond's Optimal Marginal Tax Rates

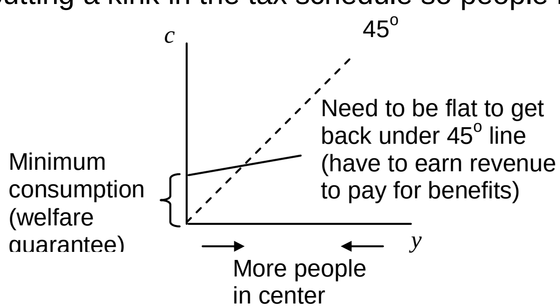


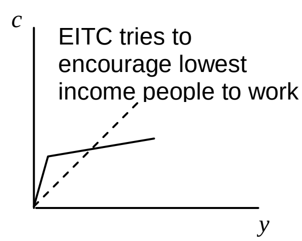
Diamond - derived optimal income tax with continuous ability distribution; had high marginal tax rates at low and high incomes (near 91%)

Why High Rates for Low Income - taking away benefits intended for low income people as their income rises is equivalent to high tax rate; giving a minimum consumption level to zero income individuals means tax schedule has to be flatter than 45^{o} (i.e., positive marginal tax rates) to get the tax schedule below the 45^{o} line so revenue is generated to pay for the benefits

Low Tax Rate Problem - based on distribution of incomes, as income rises from low income the number of people increases (just like it increases when income falls from the top earners); problem with low marginal tax rates is that the benefits would have to be given to more people (unaffordable) and those people probably don't need the benefit anyway

High Tax Rate Problem - with high marginal tax rate as in graph on left, expect people to pile up at zero income; Earned Income Tax Credit (EITC) tries to solve this by putting a kink in the tax schedule so people have to work to draw the benefits





Finite Case - recall "high ability binds" from bottom of p.3; redistributing form high to low ability results in zero marginal tax for high income and distorts low income (i.e., marginal tax rate > 0); this is common result for redistribution

Self-Selection - high marginal rates on low income in this case results from enforcing the self selection constraint on high income individuals; government can use other means to help with self-selection that may cause additional distortions (e.g., "embarrass" those who take advantage of welfare... recall food stamp vs. food ATM card discussion)

Negative Income Tax - supported by Milton Friedman; wanted to integrate welfare system into income tax (like EITC) to avoid stigma of collecting welfare

Problem - "stigma" is good for self-selection; usually a small utility cost to those who need welfare, but more severe for someone who doesn't need it

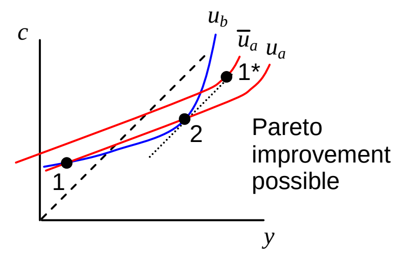
Optimal Tax - Brito, Hamilton, Stiglitz, Slutsky ("Pareto Efficient Tax Structures", 1990)

No Single Crossing - consider Stiglitz's model (pp. 2-5) over multiple periods: Will there be the same tax schedule across periods?

$V^1(c_2(1), y_2(1)) + \rho V^2(c_2(2), y_2(2))$ (assuming additive separability; person 2's utility in first period plus the discounted utility from the second period)

There is no analog to single crossing in this case so BHSS drop the assumption (similar to dropping assumption of quasiconcavity of preferences to use revealed preferences and exploiting linearity of budget constraint)

No Envy of Highest Tax Payer - consider graph; 2 good model (so we can just use two bundles to represent the tax schedule); single crossing does not apply; here both self-selection constraints bind using bundles 1 and 2, but there is a possible Pareto improvement: since type a is indifferent between 1 and 2 let him be at bundle 2 (results in more tax revenue); now we can make type b better off with 1* (no revenue cost to government)... so without single crossing constraint we get Stiglitz's original (implicit) result: person paying highest tax cannot be envied by anyone paying lower tax ("weakly envied" means indifference)



Multiple Types - m classes with N_i people in each class ($i = 1, \ldots, m$); n goods (let good one good be leisure [negative leisure is labor supply] ∴ we're looking at optimal tax combining both income and commodity taxes)

Net Trades - observe net trades, $\mathbf{x}^i \in R^n$ or each class

Utility - $U^i(\mathbf{x}^i)$; assume it's continuous and strictly monotonic (Note: we're <u>not</u> assuming quasiconcavity [equivalent of single crossing])

Producer Prices - $\mathbf{p}$; assume they're constant (not require for the results, but makes the math easier)

Tax Function - $T(\mathbf{x})$; doesn't need to be differentiable since we have distinct types (it won't be differentiable if any self selection constraints bind; see "Tax Schedule" remark at top of p.4)

Constraints -

(1) **Individual Budget Constraint** - $\mathbf{p} \cdot \mathbf{x}^i + T(\mathbf{x}^i) = 0$ (binding from monotonicity of U^i)

Linear Taxes - for optimal commodity tax, we assumed $T(\mathbf{x})$ was linear so the budget constraint was $\mathbf{q} \cdot \mathbf{x} = \mathbf{p} \cdot \mathbf{x} + \mathbf{t} \cdot \mathbf{x} = 0$... we are <u>not</u> assuming $T(\mathbf{x})$ is linear for what we're about to do

Allowable Tax Function - $T(\mathbf{x})$ is "allowable" (feasible) for each $i = 1, \ldots, m$ if $\exists\ \mathbf{x}^i \in X^i$ s.t. $\mathbf{p} \cdot \mathbf{x}^i + T(\mathbf{x}^i) \le 0$ (i.e., there exists a net trade vector in the set of feasible trades for class i such that the budget constraint it satisfied)

(2) **Government's Revenue Requirement** - $\sum_{i=1}^{m} N_i T(\mathbf{x}^i) \ge G$

Mirrlees' Trick - think of $\mathbf{x}^i$ not $T(\mathbf{x}^i)$... we'll pick bundles for consumers using their individual budget constraints so we sub $\mathbf{p} \cdot \mathbf{x} = -T(\mathbf{x}^i)$ into the government's revenue constraints: $G + \sum_{i=1}^{m} N_i (\mathbf{p} \cdot \mathbf{x}^i) \le 0$

(3) **Self selection** - $U^i(\mathbf{x}^i) \ge U^i(\mathbf{x}^j)\ \forall\ i, j$

Utility Vector - there are lots of self selection constraints and we can eliminate most of them by defining the utility vector $\mathbf{W} = (\omega_1, \omega_2, \ldots, \omega_m)$ for the net trade vector $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^m)$ where $U^i(\mathbf{x}^i) = \omega_i$; now self selection becomes: $U^i(\mathbf{x}^j) \le \omega_i \;\; \forall \;\; j \ne i$

(4) **Pareto Efficiency** - no other $\hat{\mathbf{W}}$, $\hat{\mathbf{x}}$ such that $\hat{\omega}_j \ge \omega_j \;\; \forall \;\; j$ and $\hat{\omega}_i > \omega_i$ for some i

(5) **Production Efficiency** - for now assume government is on its budget constraint (i.e., production efficiency); we'll prove this later

Min Producer Cost - set up problem to minimize producer cost of net trades:

$$\min_{\mathbf{x}} \mathbf{p} \cdot \sum_{i=1}^{m} N_i \mathbf{x}^i$$

s.t. $U^i(\mathbf{x}^i) = \omega_i \;\; \forall \;\; i$ Pareto

$U^i(\mathbf{x}^j) \le \omega_i \;\; \forall \;\; i, j \ne i$ Self Selection

Decomposes - each term in objective uses 1 type's bundle; same with each constraint so we can decompose this into m problems

$$\min_{\mathbf{x}^k} \mathbf{p} \cdot N_k \mathbf{x}^k$$

s.t. $U^k(\mathbf{x}^k) = \omega_k$ Pareto

$U^j(\mathbf{x}^k) \le \omega_j \;\; \forall \;\; j \ne k$ Self Selection

Simple Solution - if non of the self selection constraints is binding, the solution "follows directly" (pick $\mathbf{x}^k$ such that $U^k(\mathbf{x}^k) = \omega_k$)

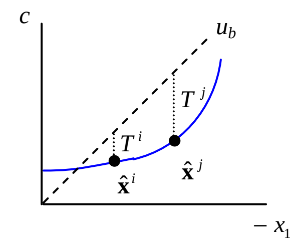
Proposition 1 - "at a Pareto efficient allocation, a group must strictly prefer its own bundle to that of a group which pays a larger total tax": if $T^b > T^a$ then $U^a(\hat{\mathbf{x}}^a) > U^a(\hat{\mathbf{x}}^b)$

Math - if $\hat{\mathbf{x}} = (\hat{\mathbf{x}}^1, \ldots, \hat{\mathbf{x}}^m)$ is constrained Pareto efficient and $U^i(\hat{\mathbf{x}}^i) = U^i(\hat{\mathbf{x}}^j)$ then $-T(\hat{\mathbf{x}}^i) = \mathbf{p} \cdot \hat{\mathbf{x}}^i \le \mathbf{p} \cdot \hat{\mathbf{x}}^j = -T(\hat{\mathbf{x}}^j)$

English - if i envies j's bundle, then j's bundle must cost at least as much i's (i.e., $\mathbf{p} \cdot \hat{\mathbf{x}}^i \le \mathbf{p} \cdot \hat{\mathbf{x}}^j$); from self selection constraints, "envies" means type i is indifferent to j's bundle: $U^i(\hat{\mathbf{x}}^i) = U^i(\hat{\mathbf{x}}^j)$

Proof: if j's bundle cost less than i's, then giving type i j's bundle would be a Pareto improvement (same utility and spend less money) so the original allocation wouldn't be optimal

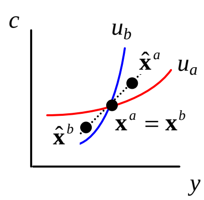
Another Way - $\mathbf{p} \cdot \hat{\mathbf{x}}^i \le \mathbf{p} \cdot \hat{\mathbf{x}}^j \Rightarrow T(\hat{\mathbf{x}}^i) \ge T(\hat{\mathbf{x}}^j)$ $\therefore$ if type i is indifferent to type j's bundle, type i must pay at least as much in taxes otherwise the government could increase revenue by moving type i to type j's bundle; recall form individual budget constraint that if type i pays less for the bundle, he pays for in taxes



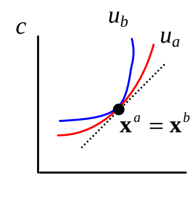
This is not an optimal solution; the government could raise more revenue by moving type i to the type j bundle, keeping type i just as happy, but raising more tax revenue; $\therefore$ optimal solution can't have indifferent types paying less in taxes

Pooling - two or more types get the same bundle; two causes

(i) Someone else envies the original bundles; (ii) all types assigned that bundle are undistorted (proposition 3)



Can make both better off and collect more revenue using $\hat{\mathbf{x}}^a$ & $\hat{\mathbf{x}}^b$ $\therefore$ if optimal solution is $\mathbf{x}^a = \mathbf{x}^b$, the better bundles violate self selection for other types



Both types a & b are undistorted at the same bundle

Self Selection Cycles - type a is indifferent to b's bundle; b is indifferent to c's bundle and c is indifferent to a's bundle: $U^a(\mathbf{x}^a) = U^a(\mathbf{x}^b)$, $U^b(\mathbf{x}^b) = U^b(\mathbf{x}^c)$, $U^c(\mathbf{x}^c) = U^c(\mathbf{x}^a)$

Proposition 2 - in a Pareto efficient allocation, there are no self selection cycles (they are broken by pooling types that are indifferent [deleting the bundle that raises less revenue])

Proposition 4 - in a Pareto efficient allocation, the group paying largest total tax is undistorted

Proposition 5 - Note: this is Len's interpretation (probably wrong): in a Pareto efficient allocation if type a is allocated $\mathbf{x}^a$ and is distorted to meet type b's self selection constraint

(i) $\mathrm{MRS}^b(\mathbf{x}^a) < \mathrm{MRS}^a(\mathbf{x}^a) < 1$ (MRT) if $x_1^a > x_1^b$ (i.e., a has less income)

(ii) (MRT) $1 < \mathrm{MRS}^a(\mathbf{x}^a) < \mathrm{MRS}^b(\mathbf{x}^a)$ if $x_1^a < x_1^b$ (i.e., a has more income)

where $x_1^i < 0$ is labor supply (negative leisure)

Same result at Stiglitz's original model (see summary on p.5)

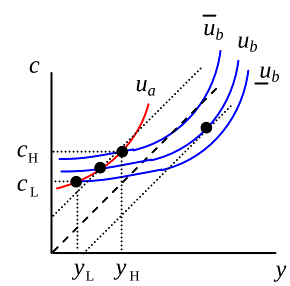
Randomization - one type gets different bundles (choose among lotteries over bundles); this pops up in incentive problems

Stiglitz - 1982 paper was original to address this; used 50-50 lottery

Reason - self selection constraints define a nonconvex set (see p.4)

Special Case - bundles in lottery only vary in consumption (same income); example would be chance of getting audited after filing income taxes (income hasn't changed, but consumption can)

General Case - suppose $V^b(c_b, y_b) = V^b(c_a, y_a)$ (type b is indifferent between b's allocation or a's allocation)



Lottery - anyone who says he's type a faces 2 bundles: $(c_{\mathrm{L}}, y_{\mathrm{L}})$ with probability $\hat{p}$ or $(c_{\mathrm{H}}, y_{\mathrm{H}})$ with probability $1-\hat{p}$; both points lie on type a's indifference curve so $V^a(c_a, y_a) = V^a(c_{\mathrm{L}}, y_{\mathrm{L}}) = V^a(c_{\mathrm{H}}, y_{\mathrm{H}})$

$\therefore$ probability is irrelevant to type a (but it does matter to get expected taxes)

Effect on Self Selection - $V^b(c_b, y_b) \geq \hat{p}V^b(c_{\mathrm{L}}, y_{\mathrm{L}}) + (1-\hat{p})V^b(c_{\mathrm{H}}, y_{\mathrm{H}})$

if type b is risk averse, he'll report the truth to avoid the lottery

When to Use - FOCs don't reveal anything; "proof is tedious... goes on for several pages"

Define $\mathbf{H}^i(c_j, y_j) = \begin{bmatrix} V_{cc}^i & V_{cy}^i \\ V_{yc}^i & V_{yy}^i \end{bmatrix}$ (evaluated at (c_j, y_j); where $V_c^i = \dfrac{\partial V^i}{\partial c}$ & $V_{cy}^i = \dfrac{\partial^2 V^i}{\partial c \partial y}$

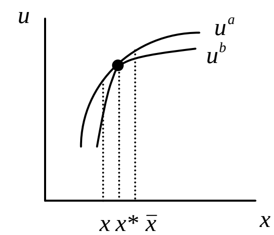
Randomization is desirable if $\dfrac{\mathbf{H}^b(c_b, y_b)}{V_c^b\left|1-\mathrm{MRS}^b\right|} - \dfrac{\mathbf{H}^a(c_b, y_b)}{V_c^a\left|1-\mathrm{MRS}^a\right|}$ is not negative semidefinite

English -

1) probabilities don't matter because we're evaluating at deterministic bundle (c_j, y_j)
2) this weird matrix is basically subtracting b's risk aversion from a's; by saying that the matrix is not negative semidefinite, we're saying b must be more risk averse than a

   Note: **risk aversion** for $U(c)$ is $-U_{cc}/U_c$

Example - think of $U^a(x)$ and $U^b(x)$ and lottery of $\underline{x}$ and $\bar{x}$; transfer of δ from b to a and want type b to consumer x^*



a's self selection: $pU^a(\underline{x}+\delta)+(1-p)U^a(\bar{x}+\delta) > U^a(x^*-\delta)$

b's self selection: $U^b(x^*-\delta) > pU^b(\underline{x}+\delta)+(1-p)U^b(\bar{x}+\delta)$

Is the lottery worth using? Yes, if $\dfrac{2U^a_{xx}}{U^a_x} - \dfrac{U^b_{xx}}{U^b_x} > 0$

(i.e., b is twice as risk averse as a)
(Hamilton didn't really go into how to derive this)

Dynamic Taxation - Brito, Hamilton, Slutsky & Stiglitz; government can link periods so instead of using $T_t(y_t)$ tax can be set up as $T_t(y_t, y_{t-1}, \ldots, y_1)$

No Commitment - government can change tax each period; supposed government uses self-selection constraints in first period to identify low and high income (ability) workers; after first period government will try to change the tax system to get as close to a lump sum tax on ability as possible

Roberts - (REStud, 1984) looked at infinite horizon; result is people won't separate in first period; worse outcome in terms of welfare than imposing self-selection constraints (can't do any redistribution)

Too Strong? - if we embed overlapping generations (new generation entering labor fore ach period), government can't tax high ability more or new generations will all pool at low ability

Full Commitment - government can charge different tax in each period, but once announced it can't change it

Discount Factor - assume government, high ability and low ability types have same discount factor ρ

Weighted Utilitarian SWF - $\max\limits_{c_a^t, y_a^t, c_b^t, y_b^t} \sum_{t=1}^{n} \rho^{t-1}\left[\alpha V^a(c_a^t, y_a^t) + (1-\alpha)V^b(c_b^t, y_b^t)\right]$

Self Selection - suppose government imposes large punishment for switching types, then we only have to worry about a single self selection constraint for each type:

$$\sum_{t=1}^{n} \rho^{t-1}\left[V^i(c_i^t, y_i^t) - V^i(c_j^t, y_j^t)\right] \geq 0 \;\; (i = a, b, \;\; j \neq i)$$

Budget Constraint - $\sum_{t=1}^{n} \rho^{t-1}\left[c_a^t - y_a^t + c_b^t - y_b^t\right] \leq 0$

Stationary Solution - repeat one period outcome

Nonstationary Solutions - analogous to randomization with fixed probabilities determined by ρ

Taxing Commodities and Income

Expand consumption to vector: $V^1(\mathbf{c}_1, y_1)$ and $V^2(\mathbf{c}_2, y_2)$ where $\mathbf{c}_i = (c_{i1}, c_{i1}, \ldots, c_{in})$

Measure consumer prices in terms of producer prices so $p_k = 1$

Pareto Problem - assuming we're transferring from type 2 to type 1

$\max V^2$

s.t. $V^1 \geq \bar{u}_1$ Pareto constraint

$V^2(\mathbf{c}_2, y_2) \geq V^2(\mathbf{c}_1, y_1)$ (λ_2) self selection constraint

$N_1 y_1 + N_2 y_2 - \sum_{k=1}^{n}(N_1 c_{1k} + N_2 c_{2k}) \geq \overline{R}$ (γ) revenue constraint

Weak Separability - $U(c, L)$ is separable in c & L

$\frac{\partial^2 U^i}{\partial c_{ij} \partial L_i} = 0 \;\forall\; i$ (types) & j (goods) $\Rightarrow$ $\frac{\partial V^1}{\partial c_{1j}} = \frac{\partial V^2}{\partial c_{2j}}$

Type 2 - completely undistorted

Labor-Commodities - $\frac{\partial V^2 / \partial c_{2j}}{\partial V^2 / \partial y} = \text{MRS}_{c_j y} = 1$ (no labor to commodity distortion)

Commodities - $\frac{\partial V^2 / \partial c_{2j}}{\partial V^2 / \partial c_{2k}} = \text{MRS}_{c_j c_k} = 1$ (no distortion among commodities)

Type 1 - undistorted among commodities, but will be distorted for labor

Commodities - $\frac{\partial V^2 / \partial c_{2j}}{\partial V^2 / \partial c_{2k}} = \text{MRS}_{c_j c_k} = \frac{N_1 \gamma + \lambda_2 \partial V^2 / \partial c_{1j}}{N_1 \gamma + \lambda_2 \partial V^2 / \partial c_{1k}} = 1$

Atkinson & Stiglitz - (1972) same result for continuous case: don't tax commodities if U is weakly separable

* this is opposite scenario as Cortlet-Hague

Deaton - (1979) don't tax commodities with optimal linear income tax if

(1) preferences are weakly separable in commodities and leisure

(2) parallel, linear Engle curves for all goods (don't need this for nonlinear income tax)

Christensen - optimal nonlinear income tax and linear taxes on commodities... used unique definition of substitutes and complements

Broadway & Keen - income tax with public goods; move away from Samuelson rule because public goods impact self selection:

$V^2(c_2, y_2, g) \geq V^2(c_1, y_1, g)$

Interactions - have to worry about interactions between y (income) and g (public good)

Example: spend money on bike trails increases MU of leisure so increasing g makes it more likely to report low ability (low income)

** Makes self selection constraint even more significant (i.e., more distortion)

Type Based Exclusions - for "public goods" that aren't (i.e., aren't non-excludable), self selection can be relaxed by using type based exclusions

Example: give away opera tickets to high ability types; idea is that low ability types who want opera tickets are really high ability misreporting

1. Determine the consequences of distributing a fixed total amount of income (Y) to maximize the following SWF:

$$W = \beta_1 y_1^\alpha + \beta_2 y_2^\alpha \quad \text{where } y_1 + y_2 = Y \text{ and } \beta_i > 0 \ (i = 1,2)$$

Consider cases:
a) $\alpha < 0$
b) $\alpha = 0$
c) $0 < \alpha < 1$
d) $\alpha \geq 1$

Setup optimization problem:

$$\max_{y_1, y_2} W = \beta_1 y_1^\alpha + \beta_2 y_2^\alpha$$

s.t. $y_1 + y_2 = Y$

The constraint is linear so second order condition is OK.
Concavity of the objective function depends on α.

Lagrangian: $L = \beta_1 y_1^\alpha + \beta_2 y_2^\alpha - \lambda(y_1 + y_2 - Y)$

(1) $\dfrac{\partial L}{\partial y_1} = \alpha\beta_1 y_1^{\alpha-1} - \lambda \leq 0$, with equality if $y_1 > 0$

(2) $\dfrac{\partial L}{\partial y_2} = \alpha\beta_2 y_2^{\alpha-1} - \lambda \leq 0$, with equality if $y_2 > 0$

(3) $-\dfrac{\partial L}{\partial \lambda} = y_1 + y_2 - Y \leq 0$, with equality if $\lambda > 0$

a) If $\alpha < 0$, then we essentially have welfare equal to each β_i divided by some power of it's respective y_i:

$$W = \frac{\beta_1}{y_1^{-\alpha}} + \frac{\beta_2}{y_2^{-\alpha}}$$

Since neither β_i nor y_i can be negative, as y_i increases W decreases. Therefore, maximizing W requires setting each y_i to **zero** (which is mathematically impossible because of division by zero).

b) $W = \beta_1 y_1^0 + \beta_2 y_2^0 = \beta_1 + \beta_2$. Therefore, W is independent of income so the distribution **doesn't matter**.

c) Assume an interior solution.
We can solve (1) and (2) for λ and set them equal to each other:

$\lambda = \alpha\beta_1 y_1^{\alpha-1} = \alpha\beta_2 y_2^{\alpha-1}$

Cancel the α and solve for the **income ratio** (equivalent to distribution):

$$\left(\frac{y_1}{y_2}\right)^{\alpha-1} = \frac{\beta_2}{\beta_1} \Rightarrow \boxed{\frac{y_1}{y_2} = \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{\alpha-1}}}$$

Since the constraint is linear, we know it will be binding. We can use the income ratio above to solve for y_1 as a function of y_2 in order to determine the amount of income each individual gets in terms of total income:

$$Y = y_1 + y_2 = y_2\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{\alpha-1}} + y_2 = \left[1+\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{\alpha-1}}\right]y_2 \Rightarrow y_2 = \frac{Y}{1+\left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{\alpha-1}}}$$

We could also solve for y_1 in terms of total income, but that's overkill (and doesn't really add much).

Check that ratio using numbers: $\beta_1 = 2$, $\beta_2 = 1$, $\alpha = 0.5$, $Y = 10$

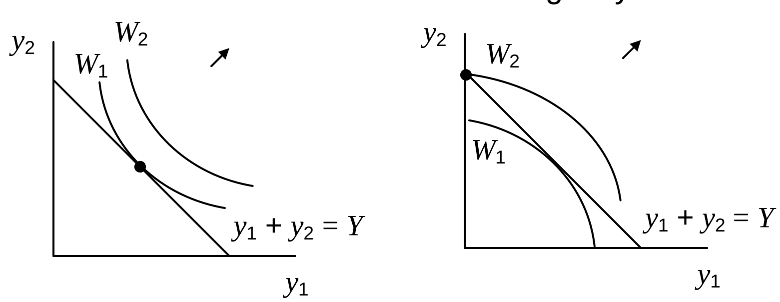
According to the ratio above, we should have $\frac{y_1}{y_2} = \left(\frac{\beta_2}{\beta_1}\right)^{\frac{1}{\alpha-1}} = \left(\frac{1}{2}\right)^{\frac{1}{0.5-1}} = 4$

Excel Solver says... $y_1 = 8$, $y_2 = 2$ $\therefore$ $\frac{y_1}{y_2} = \frac{8}{2} = 4$

d) Note in the ratio from part (c), if $\alpha = 1$ we get division by zero so we're probably not at an interior solution. In fact, at $\alpha = 1$ we have $W = \beta_1 y_1 + \beta_2 y_2$. In this case social welfare is maximized by giving **all income to a single individual** (the one with a larger β).

In the case were $\beta_1 = \beta_2$, we have the same result as part (b): income distribution **doesn't matter**.

These results also hold for $\alpha > 1$ because the objective function is not convex and we have corner solutions rather than a tangency condition.

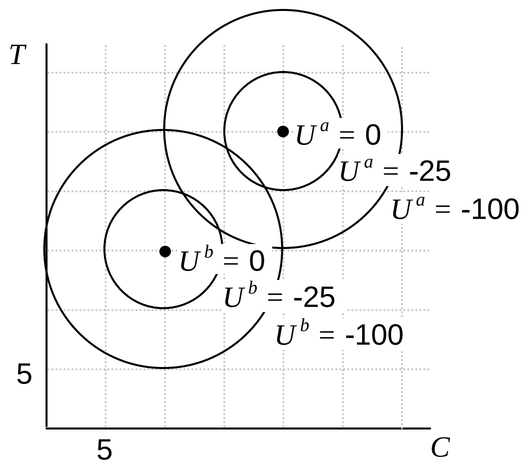


2. Consider the following economy with two people and two goods which are produced at no resource cost. Fortunately, the people become satiated with the goods, so that optima exist. The two public goods are the room temperature in degrees Celsius (T) and the number of games of cards played per day (C). Anne and Bruce are the people in this economy. Preferences are described as follows:

$$U^a(C,T) = -\left[(C-20)^2 + (T-25)^2\right]$$
$$U^b(C,T) = -\left[(C-10)^2 + (T-15)^2\right]$$

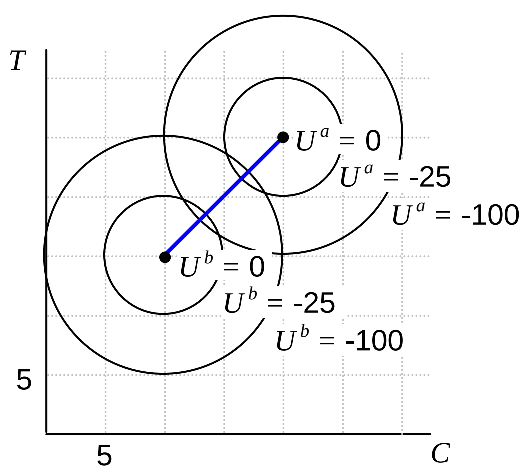
a) Sketch the indifference curves on a diagram.
b) Is $C = 10$, $T = 15$ Pareto optimal?
c) Find the set of Pareto optima. (Plane geometry may help more than solving the constrained optimization problem.)
d) Find the set of allocations that are Pareto superior to $C = 9$, $T = 14$.
e) Find a tangency between the indifference curves which is not Pareto optimal. What is going on here?

a)

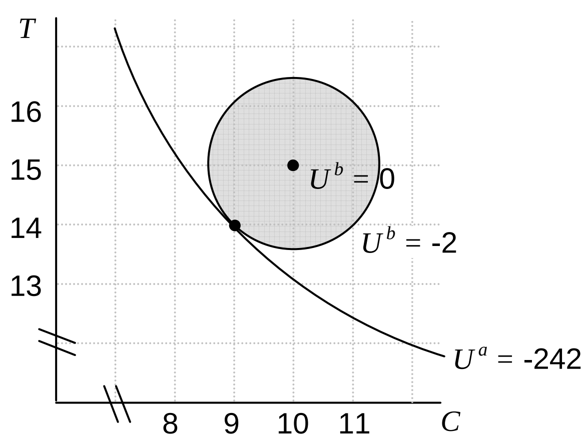


b) (10,15) is Pareto optimal because Anne cannot be made better off without making Bruce worse off.

c)



$\{(C,T) \mid T = C + 5, C \in [10,20]\}$
(the blue line in the picture)

d) 

$\{(C,T) \mid (C-10)^2 + (T-15)^2 \le 1\}$
(the shaded circle in the picture)

e) The point in (d) satisfies that condition: at (9,14) $u^b = -2$ and $u^a = -242$ are tangent, but this point is not part of the Pareto optimal set defined in (c). Preferences in this example violate the assumption of local nonsatiation so we don't have "well behaved" indifference curves and the usual tangency rule doesn't apply. "Well behaved" assumes better-than-sets don't intersect and in this case they do.

3. Supposed a social planner wishes to maximize a Bergson-Samuelson social welfare function:

$$W(U^a, U^b)$$

where $U^a = U^a\left(\mathbf{x}_a, I_a, v^b(\mathbf{x}_b)\right)$, $U^b = U^b\left(\mathbf{x}_b, I_b, v^a(\mathbf{x}_a)\right)$

$\mathbf{x}_a$ and $\mathbf{x}_b$ are the consumption vectors of a and b

and I_a and I_b are labor supplies of a and b

Assume that $\dfrac{\partial U^j}{\partial x_{ik}} = \lambda_{ij} \dfrac{\partial U^i}{\partial x_{ik}}$ for all x_{ik}, $i, j = a,b$, $i \neq j$

Note that I and j index people, while k indexes goods.
Another way to think of this condition is:

$$\frac{\partial v^i}{\partial x_{ik}} = \tilde{\lambda}_i \frac{\partial U^i}{\partial x_{ik}} \quad \text{for all } x_{ik},\ i, j = a,b,\ i \neq j$$

a) Show that the allocation which maximizes social welfare has equal marginal rates of substitution across consumers for the commodities (the x's)
b) Verify that this optimal allocation is Pareto efficient. Be careful to define Pareto efficiency in this economy.
c) Suppose marginal costs of all consumption goods are constant, labor is the only factor of production, and both individuals sell labor to buy goods at marginal cost. The individuals differ in labor endowments, utility of leisure, or productivity (or all three), so that equal consumption is not the competitive equilibrium. Will the competitive

equilibrium be Pareto efficient? If not, what government policy is needed to sustain a Pareto optimal allocation?

Define the feasible region by letting $F(\mathbf{X}, L)$ be the production function, where $\mathbf{x}_a + \mathbf{x}_b = \mathbf{X}$ and $l_a + l_b = L$. This gives the constraint:

$$F(\mathbf{X}, L) \leq 0$$

This constraint ensures that what is consumed by the two individuals can actually be produced (or purchased) with the amount of labor they can supply. Another way to do it is to define $\mathbf{Y}$ as the total amount of commodities the consumers can jointly purchase given their endowment of labor. Therefore, the constraint becomes

$$\mathbf{x}_a + \mathbf{x}_b \leq \mathbf{Y}$$
$$F(\mathbf{Y}, L) \leq 0$$

a) Maximize social welfare:

$$\max_{\mathbf{x}_a, \mathbf{x}_b, \mathbf{Y}} W(U^a, U^b) \quad \text{s.t.} \quad \mathbf{x}_a + \mathbf{x}_b = \mathbf{Y} \text{ and } F(\mathbf{Y}, L) \leq 0$$

Lagrangian: $L = W(U^a, U^b) - \boldsymbol{\mu}(\mathbf{x}_a + \mathbf{x}_b - \mathbf{Y}) - \gamma(F(\mathbf{Y}, L))$

Assume an interior solution and look at first order conditions: (corner solutions may not satisfy the equal marginal rates of substitution property)

First look at person a's consumption of commodity i:

$$\frac{\partial L}{\partial x_{ai}} = \frac{\partial W(U^a, U^b)}{\partial x_{ai}} - \mu_i = 0$$

Note: $\dfrac{\partial W(U^a, U^b)}{\partial x_{ai}} = \dfrac{\partial W}{\partial U^a} \dfrac{\partial U^a}{\partial x_{ai}} + \dfrac{\partial W}{\partial U^b} \dfrac{\partial U^b}{\partial x_{ai}}$

Substitute $\dfrac{\partial U^b}{\partial x_{ai}} = \lambda_{ab} \dfrac{\partial U^a}{\partial x_{ai}}$: $\dfrac{\partial W}{\partial x_{ai}} = \dfrac{\partial W}{\partial U^a} \dfrac{\partial U^a}{\partial x_{ai}} + \dfrac{\partial W}{\partial U^b} \lambda_{ab} \dfrac{\partial U^a}{\partial x_{ai}}$

Combine terms: $\dfrac{\partial W}{\partial x_{ai}} = \dfrac{\partial U^a}{\partial x_{ai}} \left(\dfrac{\partial W}{\partial U^a} + \lambda_{ab} \dfrac{\partial W}{\partial U^b} \right)$

So we have: $\dfrac{\partial L}{\partial x_{ai}} = \dfrac{\partial U^a}{\partial x_{ai}} \left(\dfrac{\partial W}{\partial U^a} + \lambda_{ab} \dfrac{\partial W}{\partial U^b} \right) - \mu_i = 0$

Person a's consumption of commodity j:

$$\frac{\partial L}{\partial x_{aj}} = \frac{\partial U^a}{\partial x_{aj}} \left(\frac{\partial W}{\partial U^a} + \lambda_{ab} \frac{\partial W}{\partial U^b} \right) - \mu_j = 0$$

Repeat the math for person b's consumption of commodities i and j:

$$\frac{\partial L}{\partial x_{bi}} = \frac{\partial U^b}{\partial x_{bi}} \left(\lambda_{ba} \frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b} \right) - \mu_i = 0$$

$$\frac{\partial L}{\partial x_{bj}} = \frac{\partial U^b}{\partial x_{bj}}\left(\lambda_{ba}\frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b}\right) - \mu_j = 0$$

Solve for the respective Lagrange multipliers:

$$\frac{\partial U^a}{\partial x_{ai}}\left(\frac{\partial W}{\partial U^a} + \lambda_{ab}\frac{\partial W}{\partial U^b}\right) = \mu_i \qquad \frac{\partial U^b}{\partial x_{bi}}\left(\lambda_{ba}\frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b}\right) = \mu_i$$

$$\frac{\partial U^a}{\partial x_{aj}}\left(\frac{\partial W}{\partial U^a} + \lambda_{ab}\frac{\partial W}{\partial U^b}\right) = \mu_j \qquad \frac{\partial U^b}{\partial x_{bj}}\left(\lambda_{ba}\frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b}\right) = \mu_j$$

Divide the terms in the first column to get MRS_{ij}^a

$$\frac{\frac{\partial U^a}{\partial x_{ai}}\left(\frac{\partial W}{\partial U^a} + \lambda_{ab}\frac{\partial W}{\partial U^b}\right)}{\frac{\partial U^a}{\partial x_{aj}}\left(\frac{\partial W}{\partial U^a} + \lambda_{ab}\frac{\partial W}{\partial U^b}\right)} = \frac{\frac{\partial U^a}{\partial x_{ai}}}{\frac{\partial U^a}{\partial x_{aj}}} = \frac{\mu_i}{\mu_j} = \mathrm{MRS}_{ij}^a$$

Divide the terms in the second column to get MRS_{ij}^b:

$$\frac{\frac{\partial U^b}{\partial x_{bi}}\left(\lambda_{ba}\frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b}\right)}{\frac{\partial U^b}{\partial x_{bj}}\left(\lambda_{ba}\frac{\partial W}{\partial U^a} + \frac{\partial W}{\partial U^b}\right)} = \frac{\frac{\partial U^b}{\partial x_{bi}}}{\frac{\partial U^b}{\partial x_{bj}}} = \frac{\mu_i}{\mu_j} = \mathrm{MRS}_{ij}^b$$

So $\mathrm{MRS}_{ij}^a = \mathrm{MRS}_{ij}^b$

b) Setup the Pareto problem by having person a maximize his utility given some arbitrary level of utility for person b:

$\max_{\mathbf{x}_a, \mathbf{x}_b, \mathbf{Y}} U^a$ s.t. $U^b \geq \bar{u}$, $\mathbf{x}_a + \mathbf{x}_b = \mathbf{Y}$, and $F(\mathbf{Y}, L) \leq 0$

Lagrangian: $L = U^a - \phi(U^b - \bar{u}) - \boldsymbol{\mu}(\mathbf{x}_a + \mathbf{x}_b - \mathbf{Y}) - \gamma(F(\mathbf{Y}, L))$

Assume an interior solution and look at first order conditions: (corner solutions may not satisfy the equal marginal rates of substitution property)

First look at person a's consumption of commodity i:

$$\frac{\partial L}{\partial x_{ai}} = \frac{\partial U^a}{\partial x_{ai}} - \phi\frac{\partial U^b}{\partial x_{ai}} - \mu_i = 0$$

Substitute $\frac{\partial U^b}{\partial x_{ai}} = \lambda_{ab}\frac{\partial U^a}{\partial x_{ai}}$: $\frac{\partial L}{\partial x_{ai}} = \frac{\partial U^a}{\partial x_{ai}} - \phi\lambda_{ab}\frac{\partial U^a}{\partial x_{ai}} - \mu_i = 0$

Combine terms: $\frac{\partial L}{\partial x_{ai}} = \frac{\partial U^a}{\partial x_{ai}}(1 - \phi\lambda_{ab}) - \mu_i = 0$

Person a's consumption of commodity j:

$$\frac{\partial L}{\partial x_{aj}} = \frac{\partial U^a}{\partial x_{aj}}(1-\phi\lambda_{ab}) - \mu_j = 0$$

Repeat the math for person b's consumption of commodities i and j:

$$\frac{\partial L}{\partial x_{bi}} = \frac{\partial U^b}{\partial x_{bi}}(\lambda_{ba} - \phi) - \mu_i = 0$$

$$\frac{\partial L}{\partial x_{bi}} = \frac{\partial U^b}{\partial x_{bj}}(\lambda_{ba} - \phi) - \mu_j = 0$$

Solve for the respective Lagrange multipliers:

$$\frac{\partial U^a}{\partial x_{ai}}(1-\phi\lambda_{ab}) = \mu_i \qquad \frac{\partial U^b}{\partial x_{bi}}(\lambda_{ba} - \phi) = \mu_i$$

$$\frac{\partial U^a}{\partial x_{aj}}(1-\phi\lambda_{ab}) = \mu_j \qquad \frac{\partial U^b}{\partial x_{bj}}(\lambda_{ba} - \phi) = \mu_j$$

Divide the terms in the first column to get MRS_{ij}^a

$$\frac{\frac{\partial U^a}{\partial x_{ai}}(1-\phi\lambda_{ab})}{\frac{\partial U^a}{\partial x_{aj}}(1-\phi\lambda_{ab})} = \frac{\frac{\partial U^a}{\partial x_{ai}}}{\frac{\partial U^a}{\partial x_{aj}}} = \frac{\mu_i}{\mu_j} = \mathrm{MRS}_{ij}^a$$

Divide the terms in the second column to get MRS_{ij}^b:

$$\frac{\frac{\partial U^b}{\partial x_{bi}}(\lambda_{ba} - \phi)}{\frac{\partial U^b}{\partial x_{bj}}(\lambda_{ba} - \phi)} = \frac{\frac{\partial U^b}{\partial x_{bi}}}{\frac{\partial U^b}{\partial x_{bj}}} = \frac{\mu_i}{\mu_j} = \mathrm{MRS}_{ij}^b$$

So $\mathrm{MRS}_{ij}^a = \mathrm{MRS}_{ij}^b$

c) In competitive equilibrium each person maximizes his own utility given his labor endowment without regard to the other person's utility (or consumption decision). In this case, there are externalities: one person's consumption directly affects the other person's utility. The first fundamental theorem of welfare economics which says any competitive equilibrium is also Pareto efficient does not hold in the presence of externalities. Part of the problem is that the marginal utility of income is not the same for the consumers at equilibrium. The government could put a lump-sum tax on endowments in order to get this relationship to hold in equilibrium. In this case, the consumers do not have an endowment so the lump-sum transfer would be a transfer of purchasing power. If such lump-sum transfers are not feasible, the government tries to get each person to account for the extra costs or benefits caused by their consumption decision through taxes or subsidies.

4. Consider a two-person two-good economy with the following preferences:

i) both consumers find the goods to be perfect substitutes (although not necessarily at the same rate)

ii) both consumers find the goods to be perfect complements (although not necessarily in the same proportion)

iii) one consumer finds the goods to be perfect substitutes and the other finds them to be perfect complements.

a) Find the contract curves fro these exchange economies.
b) For an arbitrary initial endowment, find the competitive equilibrium for these exchange economies.
c) For an arbitrary initial endowment, find the core of these exchange economies.
(This can all be done graphically. Be sure not to choose total levels of the goods which lead to special cases for your results.)

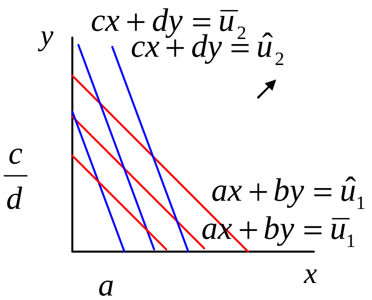
Let x and y represent the quantities of the two goods

Perfect substitutes:

$u_1 = ax + by$ and $u_2 = cx + dy$

Slopes of the indifference curves (in (x,y) space) are $-\frac{a}{b}$ and $-\frac{c}{d}$

Consumer with the steeper indifference curve prefers good y

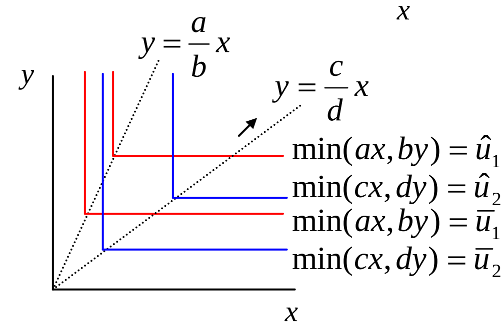


Perfect Complements:

$u_1 = \min(ax, by)$ and $u_2 = \min(cx, dy)$

Proportion in which consumers like goods depends on parameters as reflected by the corners of their indifference curves which follow lines determined by

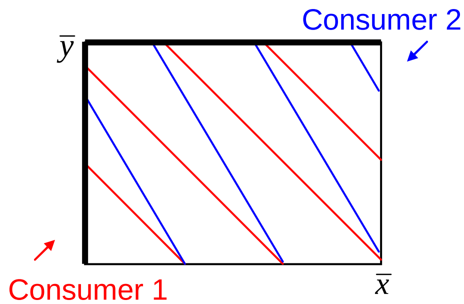
$y = \frac{a}{b}x$ and $y = \frac{c}{d}x$

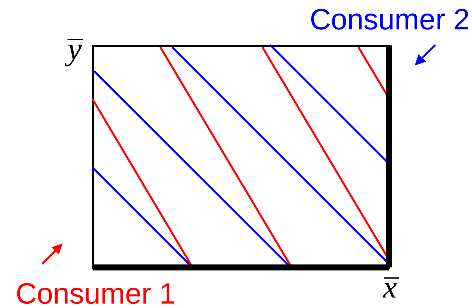


Edgeworth Box - shows consumption by both individuals where all available resources are consumed (i.e., no waste). The origin for consumer one is the lower, left corner so his utility improves as he moves to the upper, right corner. The origin for consumer two is the upper, right corner and his utility improves as he moves to the lower, left corner.

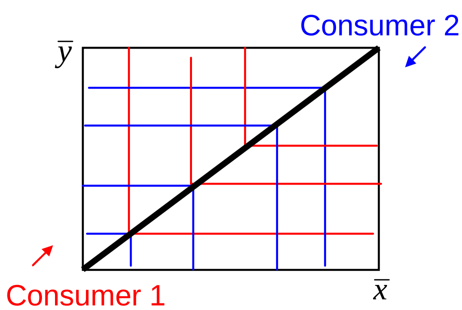
a) **Contract Curve** - set of all Pareto optimal allocations

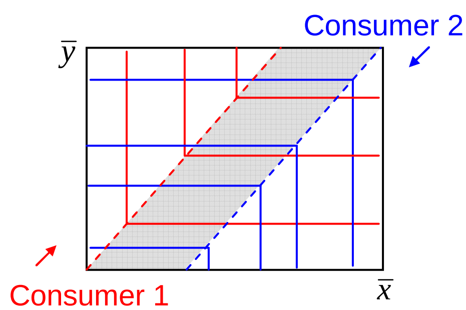
(i) Perfect Substitutes. In the case on the left, consumer 1 has the steeper indifference curves (when viewed with the same origin) so the Pareto optimal allocations favor giving good y to consumer 1 and good x to consumer 2. The case on the right is the opposite. Note, if both consumers thought the goods were perfect substitutes at the same rate, every allocation would be Pareto optimal.

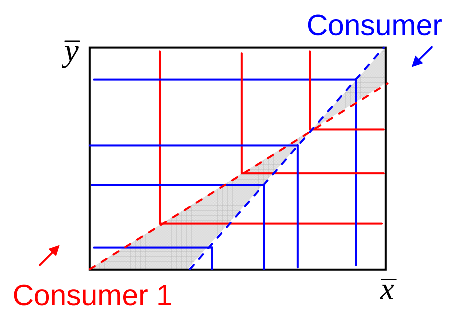




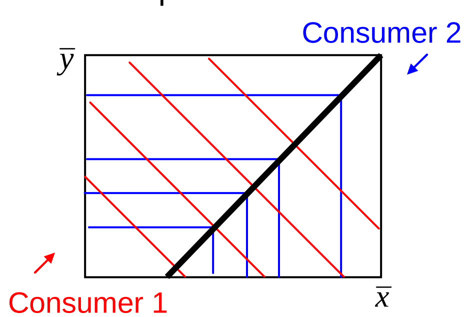
(ii) Perfect Complements. The Pareto optimal allocations are all the points between the lines that determine the proportions in which the consumers find the goods to be perfect complements (i.e., the line through the corners of all the indifference curves).



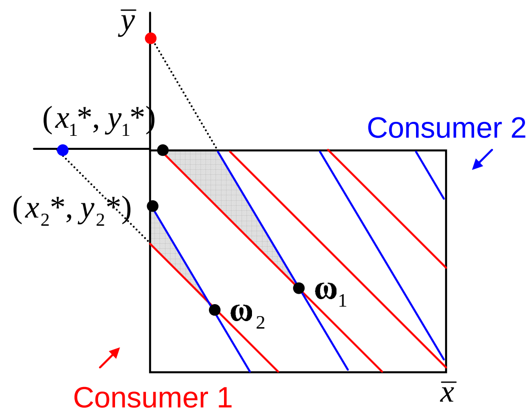


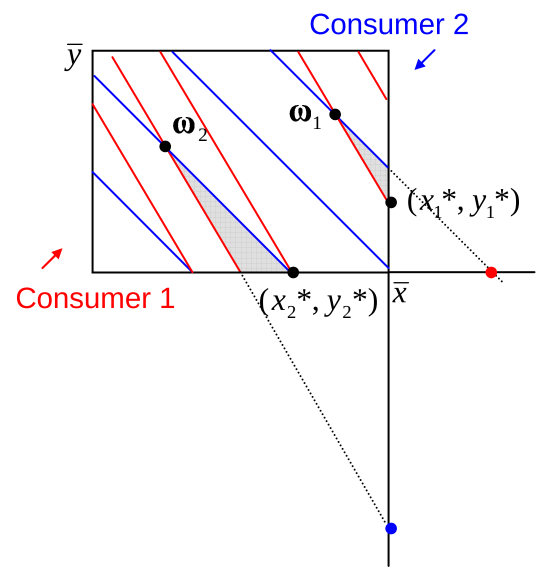


(iii) One of each. The Pareto optimal allocations are all the points on the line that determines the proportions in which the consumer who views the goods as perfect complements.



b) **Competitive Equilibrium** - allocations where both consumers maximize their utilities as price takers and there is market clearing. The gray areas show the feasible region: those allocations that are Pareto improvements (one or both consumers is better off). Given this is a well behaved problem (convex, individualistic preferences), the competitive equilibrium will be Pareto optimal. Therefore, the CE will be in the PO allocations in the shaded region (also known as the core; see part (c)). The exact equilibrium depends on the price ratio established by the two consumers. This ratio is determined by the relative preferences of the consumers, and since they are price takers, it must either put a consumer on a corner solution (i.e., zero quantity of some commodity) or a kink in the indifference curve or keep the individual indifferent to his original endowment. This will become clear in the answers below.

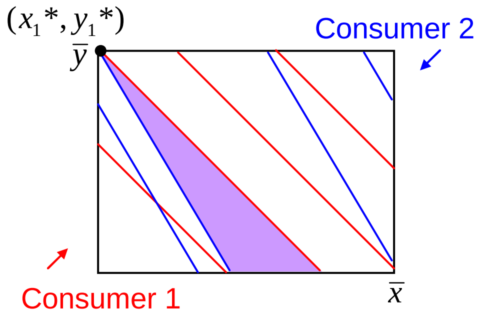


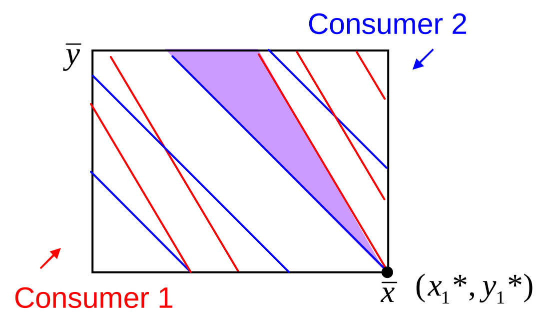


(i) Perfect Substitutes. In the top graph consumer 1 prefers good y (relative to consumer 2) so he will trade off all of his endowment of x in order to get more y. In the lower graph consumer 1 prefers good x. Given the first endowment ($\boldsymbol{\omega}_1$), if the price ratio follows consumer 2's indifference curve (the blue lines), consumer 1 will want to trade all of good x for an infeasible amount of good y. (In the lower graph he trades y for x.) The infeasible point consumer 1 wants to get to is labeled with a red dot in each graph. Now consider the price ratio being on top of consumer 1's indifference curve (the red ones). In this case, consumer 1 is indifferent between any trades. Consumer 2 trades away all of his endowment of good y (x in the lower graph) and ends up on a corner solution (zero good y in the top graph and zero good x in the bottom graph). For endowment 2, the argument is similar. If the price ratio follows consumer 1's indifference curve, consumer 2 would want to trade all of his endowment for an infeasible amount of good x (y in the lower graph; the blue dot in both graphs). If the price ratio follows consumer 2's indifference curve, consumer 1 ends up on a corner solution (zero good x on top and zero good y on bottom) and consumer 2 is indifferent.

The graphs below have purple shaded areas. Any endowments in these areas will result in the equilibria shown in the respective graphs, which are corner solutions for both consumers. The price ratio will be determined by the slope of the line through the endowment and the respective corner of the

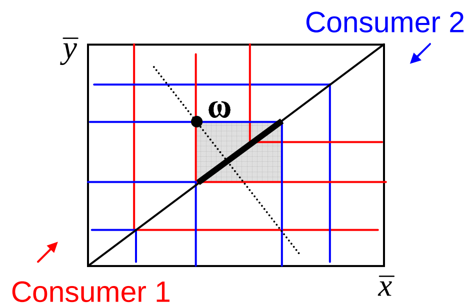
Edgeworth box. (It can vary anywhere between the slopes of the indifference curves that border the shaded area.)

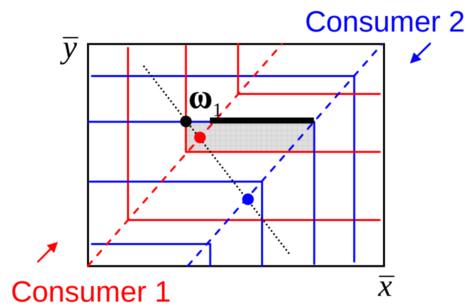


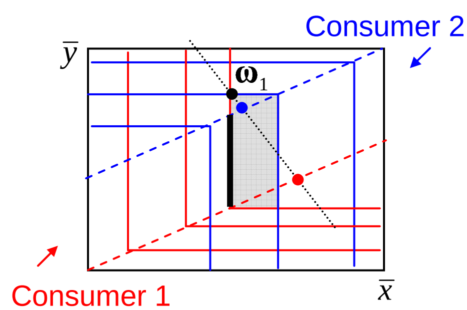


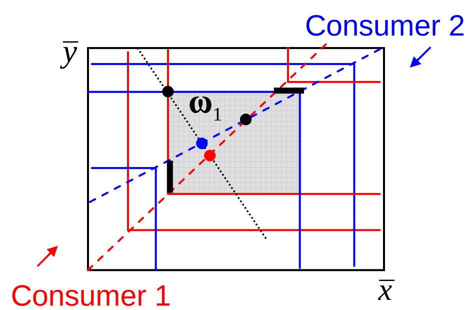
(ii) Perfect Complements. In all cases, price taking behavior for both consumers means they want to trade to the intersection of the price ratio (budget line) and the line through the corners of their indifference curves. In the first case, where the corners of their indifference curves touch, consumer 1 wants to be in the upper right corner of the feasible trades; consumer 2 in the lower left corner. At any price ratio (except $p_x = 0$ or $p_y = 0$), the consumers want to be at the same point (where the price ratio intersects their indifference curves). In the zero price cases, one of the consumers will be indifferent and the other will want to be at his optimal corner of the region of feasible trades. In other worse, there exists a competitive equilibrium for every price ratio.

The other three cases are similar (the fourth actually combines all three above it). First consider the price ratio shown (dotted black line). A blue and red dot shows where each consumer would want to be so this price ratio is not a competitive equilibrium. In the second case, only $p_x = 0$ leads to competitive equilibrium (actually multiple equilibria). In the third case, $p_y = 0$ leads to competitive equilibria as shown. The fourth cases an multiple equilibria at $p_x = 0$ and $p_y = 0$ and the price ratio determined by the line through the endowment and the point where both consumers' indifference curves are tangent (where the dashed blue and red lines intersect).

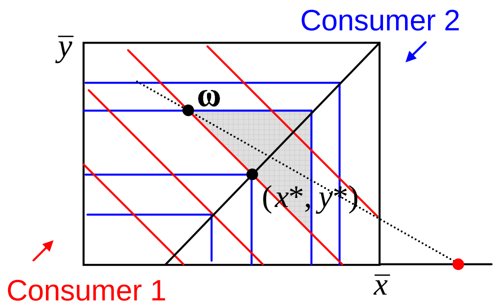






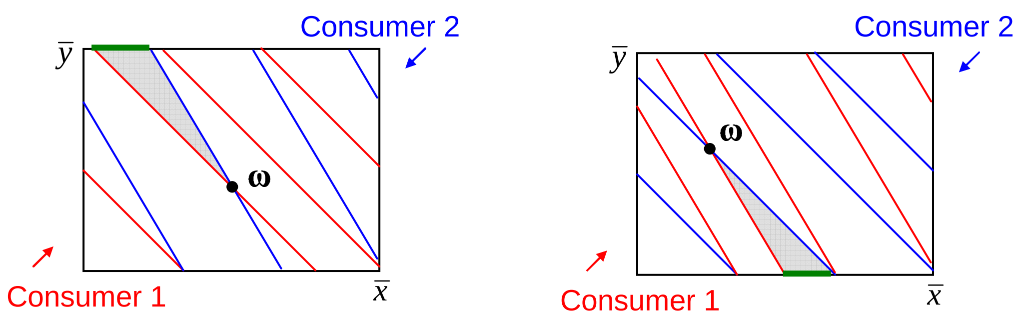


(iii) One of each. The price ratio in this case must be along consumer 1's indifference curve (the consumer who views the goods as perfect substitutes). Any other price ratio would make consumer 1 want to be outside the feasible region of trades (i.e., lower consumer 2's utility). Such a price ratio would also be physically infeasible because consumer 1 will want to trade his entire endowment of y for an amount of x that does not exist. Consumer 2 always wants to trade in such a way that he stays on the line through the corners of all his indifference curves. This is a case where "life is not fair." Consumer 1's flexibility enables trade to occur, but consumer 2 enjoys all the benefits of trade.

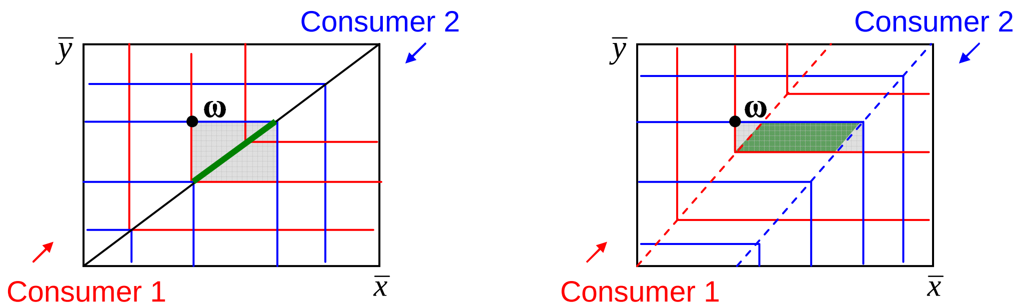


c) **Core** - collection of allocations that are attainable from a given endowment and aren't "blocked" (i.e., voted down by consumers who are made worse off). The core is found by looking at those allocations that are Pareto improvements (one or both consumers is better off) which are the allocations shaded in gray. Any points from the contract curves found in part (a) that are in this shaded area form the core (marked in green).

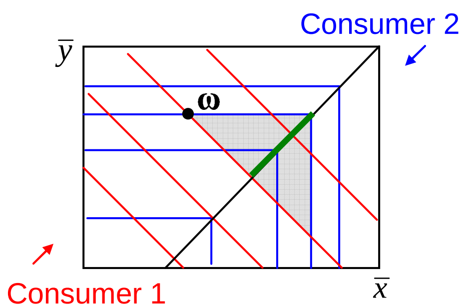
(i) Perfect Substitutes. For the case on the left, consumer 1 prefers good y so he will trade off some of his endowment of x in order to get more y. In the other case, consumer 1 trades off good y for x.



(ii) Perfect Complements. The lines connecting the corners of the indifference curves define the PO allocations as shown in part (a). The line though the intersections of the indifference curves that go through the endowment defined equilibrium prices (under the assumption of equal negotiating skill). The CE is any point where this second line crosses the set of PO allocations.



(iii) One of each. The price ratio in this case bisects the angle determined by the indifference curves that intersect at the endowment point. As before, the CE is the point where the line determined by the price ratio intersects the set of PO allocations.



Documentation.

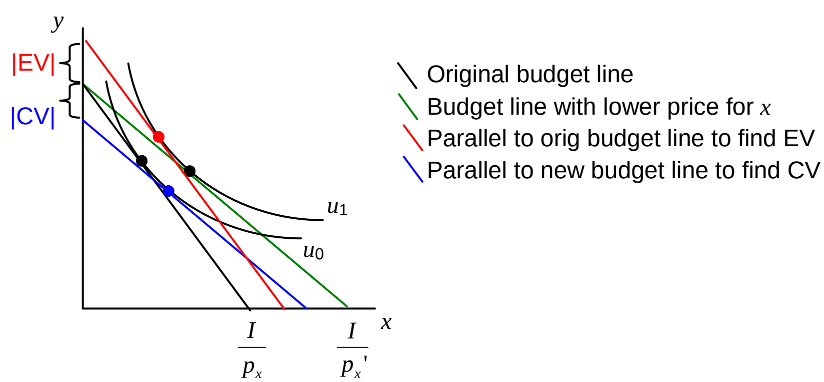
I worked out all problems with Josh Kneifel. He specifically helped with problem 1a explaining why $\alpha < 0$ leads to division by zero. For problem 3, Josh got me started with the derivative of W wrt x_{ik}. On the rest of the problems we kind of wandered aimlessly through notes and textbooks trying to figure them out together.

Prof Hamilton reviewed the competitive equilibrium graphs for problem 4c in class.

1. Suppose we want to do a rigorous cost-benefit analysis to evaluate a project which will lower the price of commodity x. Considering the argument due to Kay for distorting taxes, would you advocate using the compensating or equivalent variation measure?

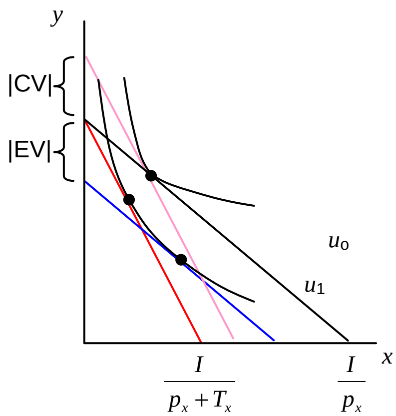
(Hamilton's Answer) We had discussed in class the infeasibility of the compensation that is implicit in the Diamond-Mirrlees use of the CV (compensating variation) based measure of excess burden. For a price decrease, it is the EV (equivalent variation) based measure that uses infeasible compensation. Look at the indifference curve diagram to see that--which one lies outside all budget lines.

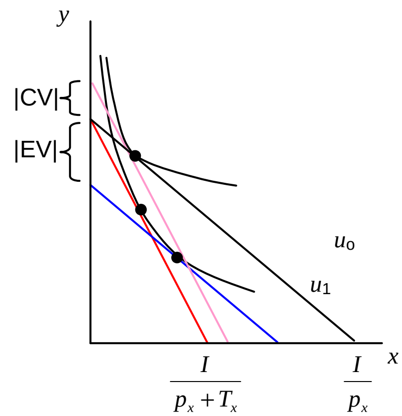
Another way to view this is that the CV and EV are inverses of each other (the CV for a price increase is the EV for the price cut). Since the EV measure is preferred for a tax increasing the price, the CV measure should be preferred for a price decrease.



2. Compare consumer surplus, equivalent variation, and compensating variation for a tax on an inferior good. Just drawing the diagrams will be enough.

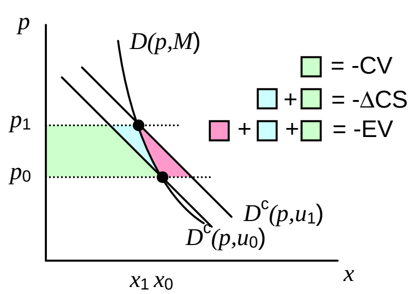
Inferior good is usually defined as one for which an increase in income lowers demand. What's actually happening is that the income effect is the opposite sign as the substitution effect. In the extreme case (Giffen good), the income effect outweighs the substitution effect so an increase in price actually increases demand (i.e., demand slopes upward). Since we are looking at compensated demands, another way to say this is that an increase in utility lowers compensated demand (i.e., if $u_0 > u_1$, $D^c(p,u_0)$ will be below (to the left of) $D^c(p,u_1)$). The result form the graphs is that CV is actually less than EV, which is the opposite of the normal good case (|EV| < |CV|). Note that in both cases (normal and inferior goods), ΔCS is still between EV and CV.

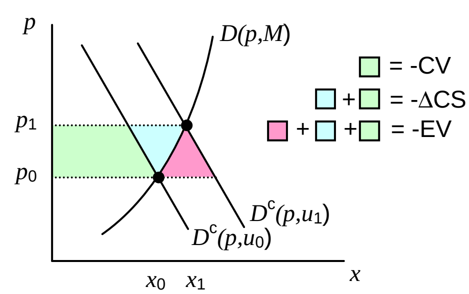




Inferior with weak income effect

Inferior with strong income effect (Giffen Good)





3. Convert the Harberger approximation for excess burden for small taxes (the double summation) to a measure which uses the own- and cross-elasticities of compensated demand. Interpret the component terms in your double summation.

Elasticity: $\varepsilon_{ij} = \dfrac{q_i}{x_j}\dfrac{\partial x_j^{c}(\mathbf{q},u_1)}{\partial q_i} \;\Rightarrow\; \dfrac{\partial x_j^{c}}{\partial q_i} = \dfrac{x_j}{q_i}\varepsilon_{ij}$

Harberger's Formula: $\bar{L} = -\dfrac{1}{2}\sum_i \sum_j t_i t_j \dfrac{\partial x_j(\mathbf{q},u_1)}{\partial q_i}$

Break up own- and cross-derivatives: $\bar{L} = -\dfrac{1}{2}\left[\sum_i t_i^2 \dfrac{\partial x_i(\mathbf{q},u_1)}{\partial q_i} + \sum_i \sum_j t_i t_j \dfrac{\partial x_j(\mathbf{q},u_1)}{\partial q_i}\right]$

Substitute elasticity: $\bar{L} = -\dfrac{1}{2}\left[\sum_i t_i^2 \dfrac{x_i}{q_i}\varepsilon_{ii} + \sum_i \sum_j t_i t_j \dfrac{x_j}{q_i}\varepsilon_{ij}\right]$

Own Effect -

$t_i^2 \geq 0$

$\dfrac{\partial x_i^{c}}{\partial q_i} = \dfrac{x_i}{q_i}\varepsilon_{ii} < 0$ (compensated demand curves always slope down)

Consider the -1/2 in front and this means the own effect of a tax is an increase in excess burden, regardless of whether the tax is positive or negative (i.e., a subsidy).

Cross Effect - the arrows below show the change in excess burden from the cross effect, not the end result of the change in excess burden. The cross effect will magnify the increase in excess burden from the own effect if both goods are taxed and they are complements or if one good is taxed and the other is subsidized and they are substitutes. (Basically, these combinations increase the distortion of the tax). Other situations shown below offset the increase in excess burden (i.e., counter the distortion) or will have no net effect because one of the goods is not taxed.

	$t_i t_j = 0$ (tax at most 1)	$t_i t_j > 0$ (tax both)	$t_i t_j < 0$ (tax one and subsidize other)
$\varepsilon_{ij} < 0$ (complements)	no change	$\bar{L}\uparrow$	$\bar{L}\downarrow$
$\varepsilon_{ij} > 0$ (substitutes)	no change	$\bar{L}\downarrow$	$\bar{L}\uparrow$

Break up tax terms: $\bar{L} = -\dfrac{1}{2}\left[\sum_i (t_i x_i)\left(\dfrac{t_i}{q_i}\right)\varepsilon_{ii} + \sum_i \sum_j (t_j x_j)\left(\dfrac{t_i}{q_i}\right)\varepsilon_{ij}\right]$

The double summation has the tax revenue of good j times the tax rate of good i (with respect to the consumer price) times the cross-elasticity of compensated demand between goods i and j. The bigger any of these terms are (in absolute value), the more likely the cross effect will magnify (or offset) the own-effect.

4. A consumer has the utility function:

$$U(x_1, x_2) = -\frac{1}{x_1} - \frac{1}{x_2}$$

let x_1 be the numeraire, and let the consumer's lump-sum income equal y.

a) Find the expenditure function.

b) What is the excess burden of a specific tax on good 2 (t_2)?

c) What is the marginal excess burden of a specific tax on good 2 (using $u = v(\mathbf{q}, y)$ where $\mathbf{q} = (1, p_2 + t_2)$ --the post-tax price vector--in the expenditure function)?

a) Consumer problem: $\max\limits_{x_1, x_2} U(x_1, x_2) = -\frac{1}{x_1} - \frac{1}{x_2}$ s.t. $\mathbf{q} \cdot \mathbf{x} = y$

Lagrangian: $\ell = -\frac{1}{x_1} - \frac{1}{x_2} - \gamma(q_1 x_1 + q_2 x_2 - y)$

$$\frac{\partial \ell}{\partial x_1} = \frac{1}{x_1^2} - \gamma q_1 = 0 \;\Rightarrow\; \gamma = \frac{1}{q_1 x_1^2}$$

$$\frac{\partial \ell}{\partial x_2} = \frac{1}{x_2^2} - \gamma q_2 = 0 \;\Rightarrow\; \gamma = \frac{1}{q_2 x_2^2}$$

Set these equal to each other: $\gamma = \frac{1}{q_1 x_1^2} = \frac{1}{q_2 x_2^2} \;\Rightarrow\; x_1 = x_2 \sqrt{\frac{q_2}{q_1}}$

Plug this into the budget constraint: $q_1 x_1 + q_2 x_2 = y$

$$q_1 x_2 \sqrt{\frac{q_2}{q_1}} + q_2 x_2 = y$$

$$x_2\left(\sqrt{q_1 q_2} + q_2\right) = y \;\Rightarrow\; \boxed{x_2 = \frac{y}{q_2 + \sqrt{q_1 q_2}}}$$

Sub into x_1:

$$x_1 = \left(\frac{y}{q_2 + \sqrt{q_1 q_2}}\right)\sqrt{\frac{q_2}{q_1}} =$$

$$\frac{y}{q_2\sqrt{\frac{q_1}{q_2}} + \sqrt{q_1 q_2}\sqrt{\frac{q_1}{q_2}}} = \frac{y}{\sqrt{q_1 q_2} + q_1} \;\Rightarrow\; \boxed{x_1 = \frac{y}{q_1 + \sqrt{q_1 q_2}}}$$

Now solve for $v(\mathbf{q}, y) = U(\mathbf{x}(\mathbf{q}, y))$:

$$-\frac{1}{x_1} - \frac{1}{x_2} = -\left(\frac{q_1 + \sqrt{q_1 q_2}}{y}\right) - \left(\frac{q_2 + \sqrt{q_1 q_2}}{y}\right) \;\Rightarrow$$

$$\boxed{v(\mathbf{q}, y) = -\frac{1}{y}\left(q_1 + q_2 + 2\sqrt{q_1 q_2}\right)}$$

To get the expenditure function use $v(\mathbf{q}, E(\mathbf{q}, u_0)) = u_0$ and solve for $E(\mathbf{q}, u_0)$

$$v(\mathbf{q}, E(\mathbf{q}, u_0)) = -\frac{1}{E(\mathbf{q}, u_0)}\left(q_1 + q_2 + 2\sqrt{q_1 q_2}\right) = u_0$$

$$\therefore \boxed{E(\mathbf{q}, u_0) = -\frac{1}{u_0}\left(q_1 + q_2 + 2\sqrt{q_1 q_2}\right)}$$

If we let good 1 be a numeraire ($q_1 = 1$), the expenditure function simplifies to:

$$\boxed{E(\mathbf{q}, u) = \frac{-1}{u}\left(1 + q_2 + 2\sqrt{q_2}\right)}$$

Other Way: (This is the "hard way" according to Prof Hamilton because we need to find $v(\mathbf{q}, y)$ later anyway.)

Expenditure function: $E(\mathbf{q}, u) = \min_{\mathbf{x}} \mathbf{q} \cdot \mathbf{x}$ s.t. $U(\mathbf{x}) \geq u$

Standard form: $E(\mathbf{q}, u) = \max_{\mathbf{x}} -\mathbf{q} \cdot \mathbf{x}$ s.t. $-U(\mathbf{x}) + u \leq 0$

Lagrangian: $L = -\mathbf{q} \cdot \mathbf{x} - \lambda(-U(\mathbf{x}) + u) = -q_1 x_1 - q_2 x_2 - \lambda\left(\frac{1}{x_1} + \frac{1}{x_2} + u\right)$

KT Conditions:

(i) $\frac{\partial L}{\partial x_1} = -q_1 + \lambda \frac{1}{x_1^2} \leq 0$ (strictly = if $x_1 > 0$)

(ii) $\frac{\partial L}{\partial x_2} = -q_2 + \lambda \frac{1}{x_2^2} \leq 0$ (strictly = if $x_2 > 0$)

(iii) $-\frac{\partial L}{\partial \lambda} = \frac{1}{x_1} + \frac{1}{x_2} + u \leq 0$ (strictly = if $\lambda > 0$)

These will all hold as equalities because we have a well behaved utility function. (There will be no corner solutions since we can't have $x_1 = 0$ or $x_2 = 0$)

Solve (i) and (ii) for λ and set them equal to each other:

$$\lambda = q_1 x_1^2 = q_2 x_2^2 \;\Rightarrow\; x_1 = x_2\sqrt{\frac{q_2}{q_1}}$$

Substitute this into (iii):

$$\frac{1}{x_2}\sqrt{\frac{q_1}{q_2}} + \frac{1}{x_2} = -u \;\Rightarrow\; x_2 = \frac{-1}{u}\left(1 + \sqrt{\frac{q_1}{q_2}}\right)$$

Solve for x_1:

$$x_1 = \frac{-1}{u}\left(1 + \sqrt{\frac{q_1}{q_2}}\right)\sqrt{\frac{q_2}{q_1}} = \frac{-1}{u}\left(1 + \sqrt{\frac{q_2}{q_1}}\right)$$

Plug x_1 and x_2 into the expenditure function:

$$E(\mathbf{q},u) = q_1\left[\frac{-1}{u}\left(1+\sqrt{\frac{q_2}{q_1}}\right)\right] + q_2\left[\frac{-1}{u}\left(1+\sqrt{\frac{q_1}{q_2}}\right)\right] = \frac{-1}{u}\left(q_1+q_2+2\sqrt{q_1q_2}\right)$$

To get the indirect utility function use $E(\mathbf{q}, v(\mathbf{q}, y)) = y$ and solve for $v(\mathbf{q}, y)$

$$E(\mathbf{q}, v(q,y)) = \frac{-1}{v(\mathbf{q},y)}\left(q_1+q_2+2\sqrt{q_1q_2}\right) = y$$

$$\therefore\ v(\mathbf{q},y) = \frac{-1}{y}\left(q_1+q_2+2\sqrt{q_1q_2}\right)$$

b) Excess burden (Kay's definition): $\bar{L} = E(\mathbf{q},u_1) - E(\mathbf{p},u_1) - (\mathbf{q}-\mathbf{p})\cdot\mathbf{x}$

Use the expenditure function from (a) and break out the tax term

$$\bar{L} = -\frac{1}{u_1}\left(q_1+q_2+2\sqrt{q_1q_2}\right) + \frac{1}{u_1}\left(p_1+p_2+2\sqrt{p_1p_2}\right) - (q_1-p_1)x_1 - (q_2-p_2)x_2$$

Since the tax is only on good 2, we have $q_1 = p_1$ and $q_2 = p_2 + t_2$

$$\bar{L} = -\frac{1}{u_1}\left(p_1 + p_2 + t_2 + 2\sqrt{p_1(p_2+t_2)}\right) + \frac{1}{u_1}\left(p_1 + p_2 + 2\sqrt{p_1p_2}\right) - t_2x_2$$

$$\bar{L} = -\frac{1}{u_1}\left(t_2 + 2\sqrt{p_1(p_2+t_2)} - 2\sqrt{p_1p_2}\right) - t_2x_2$$

$$\bar{L} = \frac{2}{u_1}\left(\sqrt{p_1p_2} - \sqrt{p_1(p_2+t_2)}\right) - t_2\left(\frac{1}{u_1} + x_2\right)$$

If we let good 1 be a numeraire ($q_1 = 1$), the formula simplifies to:

$$\bar{L} = \frac{2}{u_1}\left(\sqrt{p_2} - \sqrt{p_2+t_2}\right) - t_2\left(\frac{1}{u_1} + x_2\right)$$

Here's another way...

$$\bar{L} = \frac{y}{q_1+q_2+2\sqrt{q_1q_2}}\left(q_1+q_2+2\sqrt{q_1q_2}\right) + \frac{-y}{q_1+q_2+2\sqrt{q_1q_2}}\left(p_1+p_2+2\sqrt{p_1p_2}\right) -$$

$$(q_1-p_1)\left(\frac{y}{q_1+\sqrt{q_1q_2}}\right) - (q_2-p_2)\left(\frac{y}{q_2+\sqrt{q_1q_2}}\right)$$

$$\bar{L} = y - y\frac{p_1+p_2+2\sqrt{p_1p_2}}{q_1+q_2+2\sqrt{q_1q_2}} - (q_1-p_1)\left(\frac{y}{q_1+\sqrt{q_1q_2}}\right) - (q_2-p_2)\left(\frac{y}{q_2+\sqrt{q_1q_2}}\right)$$

Since the tax is only on good 2 and good 1 is numeraire, we have $q_1 = p_1 = 1$

$$\bar{L} = y - y\frac{1+p_2+2\sqrt{p_2}}{1+q_2+2\sqrt{q_2}} - (q_2-p_2)\frac{y}{q_2+\sqrt{q_2}}$$

$$\bar{L} = y\left[1 - \frac{1+p_2+2\sqrt{p_2}}{1+q_2+2\sqrt{q_2}} - \frac{(q_2-p_2)}{\sqrt{q_2}(1+\sqrt{q_2})}\right]$$

$$\bar{L} = y\left[\frac{1+q_2+2\sqrt{q_2}-1-p_2-2\sqrt{p_2}}{1+q_2+2\sqrt{q_2}} - \frac{(q_2-p_2)}{\sqrt{q_2}(1+\sqrt{q_2})}\right]$$

$$\bar{L} = y\left[\frac{q_2+2\sqrt{q_2}-p_2-2\sqrt{p_2}}{\left(1+\sqrt{q_2}\right)^2} - \frac{(q_2-p_2)}{\sqrt{q_2}(1+\sqrt{q_2})}\right]$$

$$\bar{L} = y\left[\frac{\sqrt{q_2}\left(q_2+2\sqrt{q_2}-p_2-2\sqrt{p_2}\right)}{\sqrt{q_2}\left(1+\sqrt{q_2}\right)^2} - \frac{(1+\sqrt{q_2})(q_2-p_2)}{\sqrt{q_2}\left(1+\sqrt{q_2}\right)^2}\right]$$

$$\bar{L} = y\left[\frac{q_2\sqrt{q_2}+2q_2-p_2\sqrt{q_2}-2\sqrt{q_2p_2}-q_2+p_2-q_2\sqrt{q_2}+p_2\sqrt{q_2}}{\sqrt{q_2}\left(1+\sqrt{q_2}\right)^2}\right]$$

$$\bar{L} = y\left[\frac{q_2-2\sqrt{q_2p_2}+p_2}{\sqrt{q_2}\left(1+\sqrt{q_2}\right)^2}\right]$$

c) Recall from (a): $v(\mathbf{q}, y) = -\frac{1}{y}\left(q_1+q_2+2\sqrt{q_1q_2}\right)$

$$\therefore \quad \frac{\partial v(\mathbf{q}, y)}{\partial q_2} = -\frac{1}{y}\left(1+\sqrt{\frac{q_1}{q_2}}\right)$$

Also from (a): $x_2 = \dfrac{y}{q_2+\sqrt{q_1q_2}}$

$$\therefore \quad \frac{\partial x_2}{\partial q_2} = \frac{-y}{\left(q_2+\sqrt{q_1q_2}\right)^2}\left(1+\frac{1}{2}\sqrt{\frac{q_1}{q_2}}\right)$$

Use $v(\mathbf{q}, y)$ for u_1 in (b) for excess burden

$$\bar{L} = -\frac{1}{v(\mathbf{q}, y)}\left(q_1+q_2+2\sqrt{q_1q_2}\right) + \frac{1}{v(\mathbf{q}, y)}\left(p_1+p_2+2\sqrt{p_1p_2}\right) - (q_1-p_1)x_1 - (q_2-p_2)x_2$$

Combine the terms with $v(\mathbf{q}, y)$ to make the derivative easier

$$\bar{L} = -\frac{1}{v(\mathbf{q}, y)}\left(q_1+q_2+2\sqrt{q_1q_2}-p_1-p_2-2\sqrt{p_1p_2}\right) - (q_1-p_1)x_1 - (q_2-p_2)x_2$$

Now take the derivative wrt q_2

$$\frac{\partial \bar{L}}{\partial q_2} = -\frac{1}{v}\left(1+\sqrt{\frac{q_1}{q_2}}\right) + \frac{1}{v^2}\frac{\partial v}{\partial q_2}\left(q_1+q_2+2\sqrt{q_1q_2}-p_1-p_2-2\sqrt{p_1p_2}\right) -$$

$$\cancel{(q_1-p_1)\frac{\partial x_1}{\partial q_2}} - \cancel{(0)x_1} - (q_2-p_2)\frac{\partial x_2}{\partial q_2} - (1)x_2$$

Realize that both of the x_1 terms drop out (the first because $q_1 = p_1$; the second because the derivative of $(q_1 - p_1)$ wrt q_2 is zero). Use the terms above for $v(\mathbf{q}, y)$, $\partial v(\mathbf{q}, y)/\partial q_2$, x_2, and $\partial x_2 / \partial q_2$

$$\frac{\partial \overline{L}}{\partial q_2} = \left(\frac{y}{q_1 + q_2 + 2\sqrt{q_1 q_2}}\right)\left(1 + \sqrt{\frac{q_1}{q_2}}\right) +$$

$$\left(\frac{y^2}{\left(q_1 + q_2 + 2\sqrt{q_1 q_2}\right)^2}\right)\left[-\frac{1}{y}\left(1 + \sqrt{\frac{q_1}{q_2}}\right)\right]\left(q_1 + q_2 + 2\sqrt{q_1 q_2} - p_1 - p_2 - 2\sqrt{p_1 p_2}\right) -$$

$$(q_2 - p_2)\left[\frac{-y}{\left(q_2 + \sqrt{q_1 q_2}\right)^2}\left(1 + \frac{1}{2}\sqrt{\frac{q_1}{q_2}}\right)\right] - \frac{y}{q_2 + \sqrt{q_1 q_2}}$$

In an attempt to make simplifying simpler, evaluate at $\mathbf{q} = (1, p_2 + t_2)$ and remember good 1 is a numeraire so $p_1 = 1$:

$$\frac{\partial \overline{L}}{\partial q_2} = \left(\frac{y}{1 + q_2 + 2\sqrt{q_2}}\right)\left(1 + \sqrt{\frac{1}{q_2}}\right) -$$

$$\left(\frac{y}{\left(1 + q_2 + 2\sqrt{q_2}\right)^2}\right)\left(1 + \sqrt{\frac{1}{q_2}}\right)\left(q_2 + 2\sqrt{q_2} - p_2 - 2\sqrt{p_2}\right) +$$

$$(q_2 - p_2)\left[\frac{y}{\left(q_2 + \sqrt{q_2}\right)^2}\left(1 + \frac{1}{2}\sqrt{\frac{1}{q_2}}\right)\right] - \frac{y}{q_2 + \sqrt{q_2}}$$

One option: factor out y so it appears the change in excess burden from a change in the price of good 2 (i.e., a tax on good 2) is proportional to the consumer's money income

$$\frac{\partial \overline{L}}{\partial q_2} = y\left[-\text{Stuff}-\right]$$

Second option: keep trying to simplify

Trick: $1 + q_2 + 2\sqrt{q_2} = \left(1 + \sqrt{q_2}\right)^2$

Trick: $1 + \sqrt{\frac{1}{q_2}} = \frac{1 + \sqrt{q_2}}{\sqrt{q_2}}$

$$\frac{\partial \overline{L}}{\partial q_2} = \left(\frac{y}{\left(1 + \sqrt{q_2}\right)^2}\right)\left(\frac{1 + \sqrt{q_2}}{\sqrt{q_2}}\right) - \left(\frac{y}{\left(1 + \sqrt{q_2}\right)^4}\right)\left(\frac{1 + \sqrt{q_2}}{\sqrt{q_2}}\right)\left(q_2 + 2\sqrt{q_2} - p_2 - 2\sqrt{p_2}\right) +$$

$$(q_2 - p_2)\left[\frac{y}{\left(q_2 + \sqrt{q_2}\right)^2}\left(1 + \frac{1}{2}\sqrt{\frac{1}{q_2}}\right)\right] - \frac{y}{q_2 + \sqrt{q_2}}$$

Trick: $\left(\frac{y}{\left(1+\sqrt{q_2}\right)^2}\right)\left(\frac{1+\sqrt{q_2}}{\sqrt{q_2}}\right)=\frac{y}{\left(q_2+\sqrt{q_2}\right)}$ (so first and forth terms cancel)

$$\frac{\partial\overline{L}}{\partial q_2}=-\left(\frac{y}{\left(1+\sqrt{q_2}\right)^4}\right)\left(\frac{1+\sqrt{q_2}}{\sqrt{q_2}}\right)\left(q_2+2\sqrt{q_2}-p_2-2\sqrt{p_2}\right)+$$

$$(q_2-p_2)\left[\frac{y}{\left(q_2+\sqrt{q_2}\right)^2}\left(1+\frac{1}{2}\sqrt{\frac{1}{q_2}}\right)\right]$$

Trick: $\left(q_2+\sqrt{q_2}\right)^2=q_2^2+2q_2\sqrt{q_2}+q_2=q_2\left(1+2\sqrt{q_2}+q_2\right)=q_2\left(1+\sqrt{q_2}\right)^2$

$$\frac{\partial\overline{L}}{\partial q_2}=-\left(\frac{y}{\left(1+\sqrt{q_2}\right)^3}\right)\left(\frac{1}{\sqrt{q_2}}\right)\left(q_2+2\sqrt{q_2}-p_2-2\sqrt{p_2}\right)+$$

$$(q_2-p_2)\left[\frac{y}{q_2\left(1+\sqrt{q_2}\right)^2}\left(1+\frac{1}{2}\sqrt{\frac{1}{q_2}}\right)\right]$$

Trick: $1+\frac{1}{2}\sqrt{\frac{1}{q_2}}=\sqrt{\frac{1}{q_2}}\left(\frac{2\sqrt{q_2}+1}{2}\right)$

$$\frac{\partial\overline{L}}{\partial q_2}=-\left(\frac{y}{\left(1+\sqrt{q_2}\right)^3}\right)\left(\frac{1}{\sqrt{q_2}}\right)\left(q_2+2\sqrt{q_2}-p_2-2\sqrt{p_2}\right)+$$

$$(q_2-p_2)\left[\frac{y}{q_2\left(1+\sqrt{q_2}\right)^2}\sqrt{\frac{1}{q_2}}\left(\frac{2\sqrt{q_2}+1}{2}\right)\right]$$

This really isn't getting any better. At one point on the board in MAT 120, we had this at

$$\frac{\partial\overline{L}}{\partial q_2}=-\frac{y}{q_2+\sqrt{q_2}}\left[\frac{q_2+2\sqrt{q_2}-p_2-2\sqrt{p_2}}{\left(1+\sqrt{q_2}\right)^4}-\frac{q_2-p_2}{q_2}\right]$$

We got excited because this turned out to have x_2 and the tax rate (wrt consumer price), but we couldn't figure out what the other term meant. At one point we managed to torture that term enough to get an x_1 in it, but that didn't make sense either.

I have no life, so I tried to use the second measure of excess burden from (b)

$$\overline{L}=y\left[\frac{q_2-2\sqrt{q_2p_2}+p_2}{\sqrt{q_2}\left(1+\sqrt{q_2}\right)^2}\right]=y\left[\frac{\sqrt{q_2}-2\sqrt{p_2}+p_2/\sqrt{q_2}}{\left(1+2\sqrt{q_2}+q_2\right)}\right]$$

$$\frac{\partial \bar{L}}{\partial q_2}=y\left[-\frac{\sqrt{q_2}-2\sqrt{p_2}+p_2/\sqrt{q_2}}{\left(1+2\sqrt{q_2}+q_2\right)^2}\left(\frac{1}{\sqrt{q_2}}+1\right)+\frac{\frac{1}{2\sqrt{q_2}}-p_2/2q_2\sqrt{q_2}}{\left(1+2\sqrt{q_2}+q_2\right)}\right]$$

$$\frac{\partial \bar{L}}{\partial q_2}=y\left[-\frac{\sqrt{q_2}-2\sqrt{p_2}+\frac{p_2}{\sqrt{q_2}}}{\left(1+\sqrt{q_2}\right)^4}\left(\frac{1}{\sqrt{q_2}}+1\right)+\frac{\frac{1}{2\sqrt{q_2}}-\frac{p_2}{2q_2\sqrt{q_2}}}{\left(1+\sqrt{q_2}\right)^2}\right]$$

$$\frac{\partial \bar{L}}{\partial q_2}=y\left[-\frac{1-2\frac{\sqrt{p_2}}{\sqrt{q_2}}+\frac{p_2}{q_2}+\sqrt{q_2}-2\sqrt{p_2}+\frac{p_2}{\sqrt{q_2}}}{\left(1+\sqrt{q_2}\right)^4}+\frac{\frac{1}{2\sqrt{q_2}}-\frac{p_2}{2q_2\sqrt{q_2}}}{\left(1+\sqrt{q_2}\right)^2}\right]$$

$$\frac{\partial \bar{L}}{\partial q_2}=y\left[-\frac{1-2\frac{\sqrt{p_2}}{\sqrt{q_2}}+\frac{p_2}{q_2}+\sqrt{q_2}-2\sqrt{p_2}+\frac{p_2}{\sqrt{q_2}}}{\left(1+\sqrt{q_2}\right)^4}+\frac{\left(\frac{1}{2\sqrt{q_2}}-\frac{p_2}{2q_2\sqrt{q_2}}\right)\left(1+2\sqrt{q_2}+q_2\right)}{\left(1+\sqrt{q_2}\right)^4}\right]$$

$$\frac{\partial \bar{L}}{\partial q_2}=y\left[\frac{-1+2\frac{\sqrt{p_2}}{\sqrt{q_2}}-\frac{p_2}{q_2}-\sqrt{q_2}+2\sqrt{p_2}-\frac{p_2}{\sqrt{q_2}}}{\left(1+\sqrt{q_2}\right)^4}+\frac{\frac{1}{2\sqrt{q_2}}+1+\frac{\sqrt{q_2}}{2}-\frac{p_2}{2q_2\sqrt{q_2}}-\frac{p_2}{q_2}-\frac{p_2}{2\sqrt{q_2}}}{\left(1+\sqrt{q_2}\right)^4}\right]$$

$$\frac{\partial \bar{L}}{\partial q_2}=\frac{y}{\left(1+\sqrt{q_2}\right)^4}\left[2\frac{\sqrt{p_2}}{\sqrt{q_2}}-2\frac{p_2}{q_2}-\frac{\sqrt{q_2}}{2}+2\sqrt{p_2}-\frac{3p_2}{2\sqrt{q_2}}+\frac{1}{2\sqrt{q_2}}-\frac{p_2}{2q_2\sqrt{q_2}}\right]$$

$$\frac{\partial \bar{L}}{\partial q_2}=\frac{y}{\left(1+\sqrt{q_2}\right)^4}\left[\frac{4q_2\sqrt{p_2}}{2q_2\sqrt{q_2}}-\frac{4\sqrt{q_2}\,p_2}{2\sqrt{q_2}\,q_2}-\frac{q_2^2}{2q_2\sqrt{q_2}}+\frac{4q_2\sqrt{q_2 p_2}}{2q_2\sqrt{q_2}}-\frac{3q_2 p_2}{2q_2\sqrt{q_2}}+\frac{q_2}{2q_2\sqrt{q_2}}-\frac{p_2}{2q_2\sqrt{q_2}}\right]$$

$$\frac{\partial \bar{L}}{\partial q_2}=\frac{y}{2q_2\sqrt{q_2}\left(1+\sqrt{q_2}\right)^4}\left[4q_2\sqrt{p_2}-4\sqrt{q_2}\,p_2-q_2^2+4q_2\sqrt{q_2 p_2}-3q_2 p_2+q_2-p_2\right]$$

I give up.

Documentation.

Prob 1. I managed this one on my own. JC made some comments that confused me, but made me reword the second paragraph.

Prob 2. Prof Hamilton clarified the part about switching the compensated demand curves. He also caught an error in the graph of a non-Giffen inferior good. He drew this same graph in class so my correction mirrors what he did.

Prob 3. I worked out the first part of this problem with Josh, Christine, and Katie (talking about changes in excess burden based on the signs of the different terms). Prof Hamilton told us in class to break up the double summation into three terms and talk about magnitudes.

Prob 4. Christine and I compared answers to part (a). We both solved the expenditure function problem directly, but then Prof Hamilton said we did it the hard way. I solved the utility maximization problem with Josh (we thought the algebra made this the hard way). Prof Hamilton said q_2 entered the excess burden equation in four places for part (c). Josh, Christine, and I wasted several hours of our lives trying to figure this out together. We literally used all 4 grease boards in Mat 120 in a futile effort that wasn't worth replicating here.

1. Consider an economy of identical individuals with preferences given by the utility function

$$U(x_1, x_2, x_3) = x_1 + \ln(x_2 - a_2) + \ln(x_3 - a_3)$$

Pre-tax prices of all three goods are normalized to one. Individuals supply good 1 (labor) ($x_1 < 0$) and consume goods 2 and 3. The government can impose *ad valorem* taxes on goods 2 and 3 at rates $\tau_2 = t_2 / p_2$ and $\tau_3 = t_3 / p_3$ to raise R dollars to meet its revenue requirements.

a) Obtain an expression for the indirect utility function as a function of the tax rates. Solve for the consumer's demands and write down the government budget constraint.

b) Find the first order necessary conditions for the optimal tax rates when $a_2 = a_3$. Are the tax rates equal?

c) Suppose $a_3 < 0$ (the consumer has an initial endowment of good 3). Do the relative tax rates depend on the sign of a_2 or on the sign of $a_2 - a_3$? Give any intuition for this result you might have.

Pre-tax prices normalized $\Rightarrow$ $p_1 = p_2 = p_3 = 1$

Taxes: $t_i = q_i - p_i$ ($\forall\ i$) and $\tau_j = t_j / p_j = t_j$ ($j = 2,3$) $\therefore$ $q_j = \tau_j + 1$

q_1 untaxed $\therefore$ $q_1 = 1$

(a) Consumer problem: $\max\limits_{x_1, x_2, x_3} U = x_1 + \ln(x_2 - a_2) + \ln(x_3 - a_3)$ s.t. $\mathbf{q} \cdot \mathbf{x} = 0$

Lagrangian: $\ell = x_1 + \ln(x_2 - a_2) + \ln(x_3 - a_3) - \gamma(q_1 x_1 + q_2 x_2 + q_3 x_3)$

FOCs: $U(x_1, x_2, x_3)$ guarantees $x_2, x_3 > 0$; given $x_1 < 0$ $\therefore$ interior solution

$$\frac{\partial \ell}{\partial x_1} = 1 - \gamma q_1 = 0 \;\;\Rightarrow\;\; \gamma = \frac{1}{q_1} \qquad [1]$$

$$\frac{\partial \ell}{\partial x_2} = \frac{1}{x_2 - a_2} - \gamma q_2 = 0 \;\;\Rightarrow\;\; \gamma = \frac{1}{q_2 (x_2 - a_2)} \qquad [2]$$

$$\frac{\partial \ell}{\partial x_3} = \frac{1}{x_3 - a_3} - \gamma q_3 = 0 \;\;\Rightarrow\;\; \gamma = \frac{1}{q_3 (x_3 - a_3)} \qquad [3]$$

$$-\frac{\partial \ell}{\partial \gamma} = q_1 x_1 + q_2 x_2 + q_3 x_3 = 0 \qquad [4]$$

Sub [1] using $q_1 = 1$ into [2] and [3]; solve for x_2 and x_3:

$$1 = \frac{1}{q_2 (x_2 - a_2)} \;\;\Rightarrow\;\; q_2 (x_2 - a_2) = 1 \;\;\Rightarrow\;\; x_2 = \frac{1 + a_2 q_2}{q_2} = \frac{1}{q_2} + a_2$$

$$1 = \frac{1}{q_3 (x_3 - a_3)} \;\;\Rightarrow\;\; q_3 (x_3 - a_3) = 1 \;\;\Rightarrow\;\; x_3 = \frac{1 + a_3 q_3}{q_3} = \frac{1}{q_3} + a_3$$

Sub these into [4]; solve for x_1:

$$(1)x_1 + q_2\left(\frac{1+a_2q_2}{q_2}\right) + q_3\left(\frac{1+a_3q_3}{q_3}\right) = 0$$

$$x_1 + (1+a_2q_2) + (1+a_3q_3) = 0 \;\;\Rightarrow\;\; x_1 = -(2+a_2q_2+a_3q_3)$$

Sub $q_j = \tau_j + 1$ ($j = 2,3$) to get consumer demands in terms of tax rates:

$$\boxed{\begin{aligned} x_1(\tau_2,\tau_3) &= -(2+a_2(\tau_2+1)+a_3(\tau_3+1)) \\ x_2(\tau_2,\tau_3) &= \frac{1}{\tau_2+1} + a_2 \\ x_3(\tau_2,\tau_3) &= \frac{1}{\tau_3+1} + a_3 \end{aligned}}$$

Solve for $V(\tau_2,\tau_3) = U(\mathbf{x}(\tau_2,\tau_3))$ to get

$$V(\tau_2,\tau_3) = -(2+a_2(\tau_2+1)+a_3(\tau_3+1)) + \ln\left(\frac{1}{\tau_2+1} + a_2 - a_2\right) + \ln\left(\frac{1}{\tau_3+1} + a_3 - a_3\right)$$

$$\boxed{V(\tau_2,\tau_3) = -(2+a_2(\tau_2+1)+a_3(\tau_3+1)) + \ln\left(\frac{1}{\tau_2+1}\right) + \ln\left(\frac{1}{\tau_3+1}\right)}$$

Government Budget Constraint:

$$\tau_2 x_2 + \tau_3 x_3 \geq R$$

Plug in equilibrium values for x_2 and x_3:

$$\boxed{\tau_2\left(\frac{1}{\tau_2+1} + a_2\right) + \tau_3\left(\frac{1}{\tau_3+1} + a_3\right) \geq R}$$

(b) $$\max_{\tau_2,\tau_3} V(\tau_2,\tau_3) = -(2+a_2(\tau_2+1)+a_3(\tau_3+1)) + \ln\left(\frac{1}{\tau_2+1}\right) + \ln\left(\frac{1}{\tau_3+1}\right)$$

s.t. $\tau_2\left(\frac{1}{\tau_2+1} + a_2\right) + \tau_3\left(\frac{1}{\tau_3+1} + a_3\right) \geq R$

Lagrangian:

$$L = -(2+a_2(\tau_2+1)+a_3(\tau_3+1)) + \ln\left(\frac{1}{\tau_2+1}\right) + \ln\left(\frac{1}{\tau_3+1}\right) -$$

$$\lambda\left[\tau_2\left(\frac{1}{\tau_2+1} + a_2\right) + \tau_3\left(\frac{1}{\tau_3+1} + a_3\right) - R\right]$$

FOCs:

$$\frac{\partial L}{\partial \tau_2} = -a_2 - \frac{1}{\tau_2+1} - \lambda\left[\frac{1}{\tau_2+1} + a_2 - \frac{\tau_2}{(\tau_2+1)^2}\right] = 0$$

$$\frac{\partial L}{\partial \tau_3} = -a_3 - \frac{1}{\tau_3+1} - \lambda\left[\frac{1}{\tau_3+1} + a_3 - \frac{\tau_3}{(\tau_3+1)^2}\right] = 0$$

$$-\frac{\partial L}{\partial \lambda} = \tau_2\left(\frac{1+a_2(\tau_2+1)}{\tau_2+1}\right) + \tau_3\left(\frac{1+a_3(\tau_3+1)}{\tau_3+1}\right) - R = 0$$

Take FOC for τ_2; pull a_2 out of brackets and get common denominator

$$\frac{\partial L}{\partial \tau_2} = -a_2 - \frac{1}{\tau_2+1} - \lambda a_2 - \lambda\left[\frac{\tau_2+1}{(\tau_2+1)^2} - \frac{\tau_2}{(\tau_2+1)^2}\right] = 0$$

Combine a_2 terms; get common denominator for second and third terms

$$-(1+\lambda)a_2 - \frac{\tau_2+1}{(\tau_2+1)^2} - \frac{\lambda}{(\tau_2+1)^2} = 0$$

Solve for a_2

$$-(1+\lambda)a_2 = \frac{\tau_2+1+\lambda}{(\tau_2+1)^2} \Rightarrow a_2 = \frac{\tau_2+1+\lambda}{-(1+\lambda)(\tau_2+1)^2}$$

FOC for a_3 mirrors this derivation: $a_3 = \dfrac{\tau_3+1+\lambda}{-(1+\lambda)(\tau_3+1)^2}$

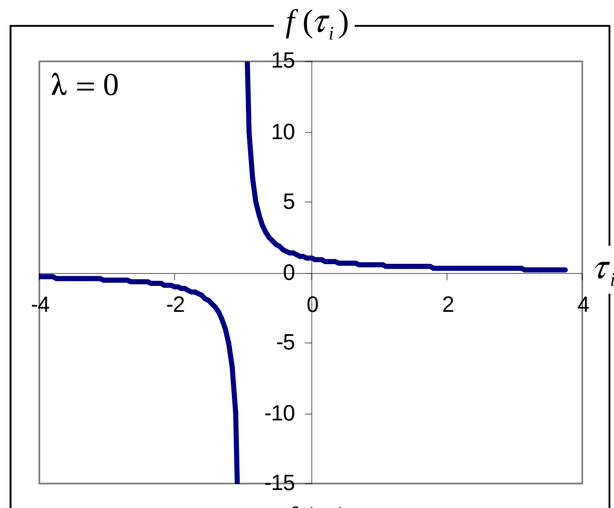
If we have $a_2 = a_3$, these terms must be equal:

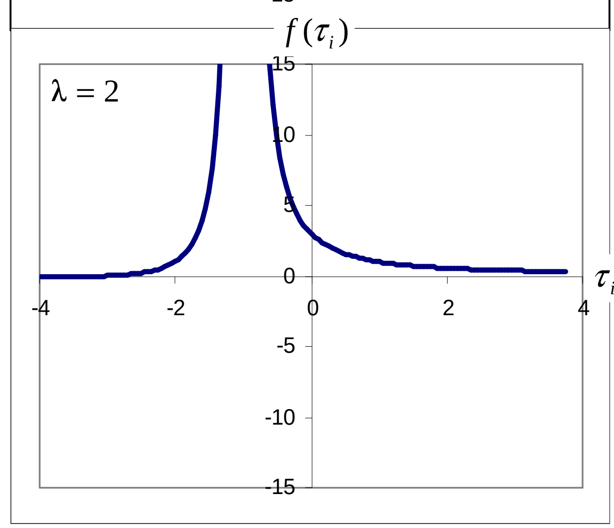
$$\frac{\tau_2+1+\lambda}{-(1+\lambda)(\tau_2+1)^2} = \frac{\tau_3+1+\lambda}{-(1+\lambda)(\tau_3+1)^2}$$

Cancel the $-(1+\lambda)$ from both terms

$$\frac{\tau_2+1+\lambda}{(\tau_2+1)^2} = \frac{\tau_3+1+\lambda}{(\tau_3+1)^2}$$

$$f(\tau_2) = f(\tau_3)$$





Rather than make fancy mathematical arguments, I plugged this into Excel using $\lambda = 0$ and $\lambda = 1$. The graph with $\lambda = 0$ shows the only way $f(\tau_2) = f(\tau_3)$ is for $\tau_2 = \tau_3$, however for any $\lambda > 0$ it is possible to have $f(\tau_2) = f(\tau_3)$ with $\tau_2 \neq \tau_3$. In these situations, the tax rates will have opposite signs.

$\therefore$ if $\lambda = 0$ (the budget constraint is not binding) or if $\lambda > 0$ and the tax rates have the same sign, then the **tax rates are equal**. (In the remaining case, $\lambda > 0$ and the tax rates have different signs, obviously the tax rates are different.)

(c) Looking at the consumer's demands from part (a), if $a_3 < 0$ then x_3 is most likely negative (doesn't have to be, but it will be less than if $a_3 > 0$). This frees up resources in the consumer's budget constraint which, based on the consumer's objective function will be put to better use on good 2, assuming $a_2 > a_3$. The point at which $MRS_{1,2} = MRS_{1,3}$ will be determined by the difference between a_2 and a_3. This will determine the demands which then will drive the tax rates in order to satisfy the government's budget constraint. Therefore, the relative tax rates depend on the sign of $\boldsymbol{a_2 - a_3}$.

2. Consider Diamond and Mirrlees's AER papers on optimal commodity taxation. Optimal taxation and productive efficiency are linked together as principles of intervention. Suppose, unlike the traditional model, that both goods 1 and 2 are untaxable--as a prior restriction.
a) To keep it simple, consider a one-consumer economy with 4 goods ($q_1 = p_1 = 1$, $q_2 = p_2$ from the tax restrictions). Start from their original problem (equation (15)) and work through to get an optimization problem analogous to (17). What impact does the constraint that $t_2 = 0$ have?
b) Show that productive efficiency is not desirable in this economy.

(a) Original problem:

$$(15)\quad \begin{cases} \max\limits_{\substack{q_3,q_4\\ p_3,p_4\\ z_1,z_2,z_3,z_4}} V(q_1,q_2,q_3,q_4) & \\ \text{s.t. } x_i(\mathbf{q}) - y_i - z_i = 0,\ i = 1,2,3,4 & \text{(1) market clearing} \\ \mathbf{y} = \arg\max\limits_{\mathbf{p}} \mathbf{p}\cdot\mathbf{y} \text{ s.t. } y_1 = f(y_2,y_3,y_4) & \text{(2) private production efficiency} \\ z_1 = g(z_2,z_3,z_n) & \text{(3) public production efficiency} \\ q_2 = p_2 & \text{(4) no tax on good 2} \end{cases}$$

Use market clearing to solve for private production quantities:

$$y_i = x_i(\mathbf{q}) - z_i = 0,\ i = 2,3,4$$

Sub (2) and (3) into market clearing constraint for good 1:

$$x_1(\mathbf{q}) = y_1 + z_1 = f(y_2,y_3,y_4) + g(z_2,z_3,z_n)$$

Sub the values for y_2, y_3, y_4 from the first step:

$$x_1(\mathbf{q}) = f(x_2(\mathbf{q}) - z_2, x_3(\mathbf{q}) - z_3, x_4(\mathbf{q}) - z_4) + g(z_2,z_3,z_n)$$

Private production efficiency FOCs:

$$p_1 - \lambda = 0 \text{ and } p_i - \lambda\frac{\partial f}{\partial y_i} = 0\ (i = 2,3,4)$$

Let $f_i = \dfrac{\partial f}{\partial y_i}$ so we have $p_i = p_1 f_i$ $(i = 2,3,4)$

$\therefore$ $q_2 = p_2$ becomes $q_2 = f_2$

New problem:

$$(17)\quad \begin{cases} \max\limits_{\substack{q_2,q_3,q_4\\ z_2,z_3,z_4}} V(\mathbf{q}) \text{ s.t.} \\ \text{s.t. } x_1(\mathbf{q}) = f(x_2(\mathbf{q}) - z_2, x_3(\mathbf{q}) - z_3, x_4(\mathbf{q}) - z_4) + g(z_2, z_3, z_n) \\ \quad q_2 = f_2 \end{cases}$$

The restriction that $t_2 = 0$ means there is an additional constraint $q_2 = f_2$ which essentially eliminates a decision variable.

(b) Lagrangian:

$$\ell = V(\mathbf{q}) - \lambda[x_1(\mathbf{q}) - f(x_2(\mathbf{q}) - z_2, x_3(\mathbf{q}) - z_3, x_4(\mathbf{q}) - z_4) + g(z_2, z_3, z_n)] - \mu[q_2 - f_2]$$

FOCs:

$$\frac{\partial \ell}{\partial z_k} = -\lambda[f_k - g_k] - \mu \frac{\partial f_2}{\partial z_k} = 0$$

Assuming an interior solution, we can solve for f_k : $f_k = g_k - \dfrac{\mu}{\lambda}\dfrac{\partial f_2}{\partial z_k}$

If we assume the private production function f is twice differentiable and nonlinear in y (so the $\partial f_2 / \partial z_k$ exists and is $\neq 0$), then $f_k \neq g_k$ (marginal rate of transformation in private and public sectors are not equal) so we do not have aggregate production efficiency.

3. Go back to Kay's paper and derive the optimal tax formulas by minimizing excess burden where the utility level in the expenditure function is indirect utility $v(\mathbf{q},0)$. Explain why using the compensating variation in this approach will not yield the same results.

Excess burden using equivalent variation: $\bar{L} = E(\mathbf{q}, u_1) - E(\mathbf{p}, u_1) - (\mathbf{q} - \mathbf{p}) \cdot \mathbf{x}(\mathbf{q})$

Substitute indirect utility: $\bar{L} = E(\mathbf{q}, v(\mathbf{q},0)) - E(\mathbf{p}, v(\mathbf{q},0)) - \mathbf{t} \cdot \mathbf{x}(\mathbf{q})$

Assume pre-tax prices ($\mathbf{p}$) are fixed (Hamilton told us to)

Assume government budget constraint is equality (like Kay did)

Problem: $\min\limits_{\mathbf{q}} \bar{L} = E(\mathbf{q}, v(\mathbf{q},0)) - E(\mathbf{p}, v(\mathbf{q},0)) - (\mathbf{q} - \mathbf{p}) \cdot \mathbf{x}(\mathbf{q})$ s.t. $(\mathbf{q} - \mathbf{p}) \cdot \mathbf{x}(\mathbf{q}) = R$

Lagrangian: $\ell = E(\mathbf{q}, v(\mathbf{q},0)) - E(\mathbf{p}, v(\mathbf{q},0)) - (\mathbf{q} - \mathbf{p}) \cdot \mathbf{x}(\mathbf{q}) + \lambda((\mathbf{q} - \mathbf{p}) \cdot \mathbf{x}(\mathbf{q}) - R)$

Note: setting it up with $+\lambda$ rather than the standard $-\lambda$ in order to get the Diamond and Mirrlees optimal commodity tax rule

FOCs:

$$\frac{\partial \ell}{\partial q_k} = \frac{\partial E(\mathbf{q}, v(\mathbf{q},0))}{\partial q_k} + \frac{\partial E(\mathbf{q}, v(\mathbf{q},0))}{\partial v}\frac{\partial v(\mathbf{q},0)}{\partial q_k} - \frac{E(\mathbf{p}, v(\mathbf{q},0))}{\partial v}\frac{\partial v(\mathbf{q},0)}{\partial q_k}$$

$$- x_k(\mathbf{q}) - \sum_{i=1}^{n}(q_i - p_i)\frac{\partial x_i(\mathbf{q})}{\partial q_k} + \lambda\left(x_k(\mathbf{q}) + \sum_{i=1}^{n}(q_i - p_i)\frac{\partial x_i(\mathbf{q})}{\partial q_k}\right) = 0$$

Note (courtesy of micro notes): $\dfrac{\partial E(\mathbf{q}, v(\mathbf{q},0))}{\partial q_k} = x_k^c(\mathbf{q}, v(\mathbf{q},0)) = x_k(\mathbf{q})$

$$\sum_{i=1}^{n}(q_i - p_i)\frac{\partial x_i(\mathbf{q})}{\partial q_k} = \lambda\left(x_k(\mathbf{q}) + \sum_{i=1}^{n}(q_i - p_i)\frac{\partial x_i(\mathbf{q})}{\partial q_k}\right)$$

Sub in $t_i = q_i - p_i$ and this is the same as (3") on p.115 of the Kay article:

$$\boxed{\sum_{i=1}^{n} t_i \frac{\partial x_i(\mathbf{q})}{\partial q_k} = \lambda\left(x_k(\mathbf{q}) + \sum_{i=1}^{n} t_i \frac{\partial x_i(\mathbf{q})}{\partial q_k}\right)}$$

Excess burden using compensating variation and indirect utility:

$$\tilde{L} = E(\mathbf{q}, v(\mathbf{p},0)) - E(\mathbf{p}, v(\mathbf{p},0)) - (\mathbf{q}-\mathbf{p})\cdot\mathbf{x}(\mathbf{q})$$

Problem: $\min_{\mathbf{q}} \tilde{L} = E(\mathbf{q}, v(\mathbf{p},0)) - E(\mathbf{p}, v(\mathbf{p},0)) - (\mathbf{q}-\mathbf{p})\cdot\mathbf{x}(\mathbf{q})$ s.t. $(\mathbf{q}-\mathbf{p})\cdot\mathbf{x}(\mathbf{q}) = R$

Lagrangian: $\tilde{\ell} = E(\mathbf{q}, v(\mathbf{p},0)) - E(\mathbf{p}, v(\mathbf{p},0)) - (\mathbf{q}-\mathbf{p})\cdot\mathbf{x}(\mathbf{q}) - \gamma\big((\mathbf{q}-\mathbf{p})\cdot\mathbf{x}(\mathbf{q}) - R\big)$

FOCs:

$$\frac{\partial\tilde{\ell}}{\partial q_k} = \frac{\partial E(\mathbf{q}, v(\mathbf{p},0))}{\partial q_k} - x_k(\mathbf{q}) - \sum_{i=1}^{n}(q_i - p_i)\frac{\partial x_i(\mathbf{q})}{\partial q_k} - \gamma\left(x_k(\mathbf{q}) + \sum_{i=1}^{n}(q_i - p_i)\frac{\partial x_i(\mathbf{q})}{\partial q_k}\right)$$

This time $\dfrac{\partial E(\mathbf{q}, v(\mathbf{p},0))}{\partial q_k} = x_k^{c}(\mathbf{q}, v(\mathbf{p},0)) \neq x_k(\mathbf{q})$

Using compensated variation does not yield the same result because the indirect utility is evaluated at pre-tax prices and will not change with respect to post-tax prices (i.e., they're independent of the tax).

Documentation.

Prob 1. I set up part (a) on my own and got confirmation from everyone that it was correct. Josh showed me a better way to simplify the demands to make part (b) easier. JC and I tried to get a solution using Mathematica, but it didn't work. The trick in part (b) of solving for a_2 and a_3 came from Nick (by way of Christine and Katie). I talked to everyone about part (c), but never really felt comfortable with what I heard. I tried various numerical solutions in Excel, but the problem is very unstable and small changes in parameters caused huge changes so I wasn't able to get a "feel" for what was going on.

Prob 2. Prof. Hamilton confirmed the basics of the problem (going from (15) to (17) was essentially unchanged and we just have a new constraint). Christine pointed out that the aggregate production efficiency result came from the derivative wrt z_k. I checked my results in JC and Josh.

Prob 3. Christine and Prof. Hamilton both pointed me to the optimal tax formulas (3") in the Kay article. Prof. Hamilton said to use regular demands for all the tax revenue computations. He practically did the whole problem for us because we bugged him with so many questions.